Orthotropic Plates:
Fundamentals of Theory and Simplified Solution Methods

Short notes

INSA – Rennes, January/February 2009

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1. Theoretical formulation

The present short notes explain the basic features of the Theory of Thin Plates with particular emphasis to orthotropic ones.

A final application of an approximate procedure based on the double Fourier series is also proposed and applied.

Far from being a complete and exhaustive textbook on the theory of plates (which can be rather found within the texts mentioned in the last chapter) the present notes are only intended as a short guide for remarking the basic assumptions and the final results of the theory and supporting students in finding approximate solutions by using double Fourier series.

1.1 Introduction and preliminary definitions

A plate can be defined as a three-dimensional body characterized by the two basic geometric properties commented below [2]:

- Thinness: one of the plate dimensions, its *thickness*, is much smaller than the other two ones;
- Flatness: the midsurface of the plate, which is the locus of the points that halve the thickness "fibers" or "filaments", is a plane.

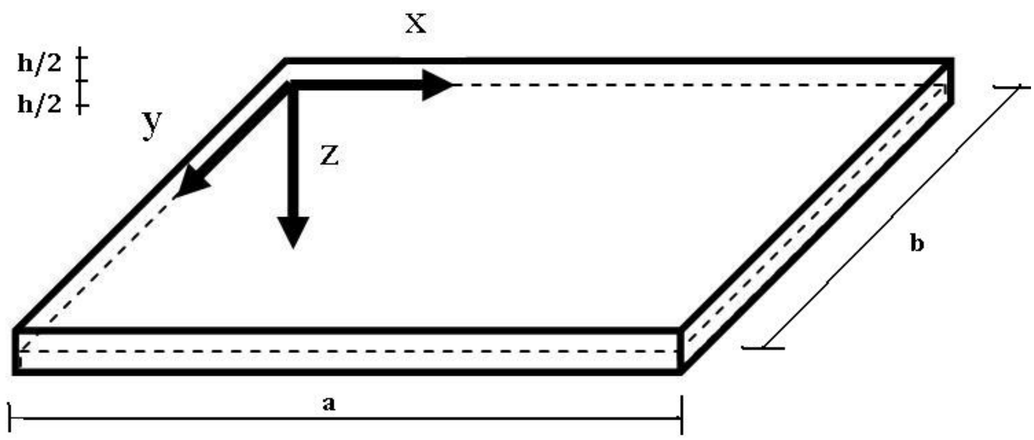


Figure 1: rectangular thin plate

The bending problem in plates is completely described by the displacement field w(x,y) of the above mentioned mid surface:

$$w = w(x,y) \ . \tag{1}$$

The other displacement components are zero on the midsurface (at least in the considered case of only transverse load applied on the plate) and can be easily derived depending as a function of w(x,y) as the following hypothesis, usually assumed for "thin plates", is adopted:

> *the general chord of the plate (obtained by intersecting the body with two non-parallel planes orthogonal to the mid surface) remains perpendicular to the mid surface in the deformed configuration.*

1.2 Displacement field

On the basis of the above kinematic assumptions briefly outlined above, the overall displacement field of the body can be easily derived as shown in the following subsections.

1.2.1 Displacements in the x-z plane

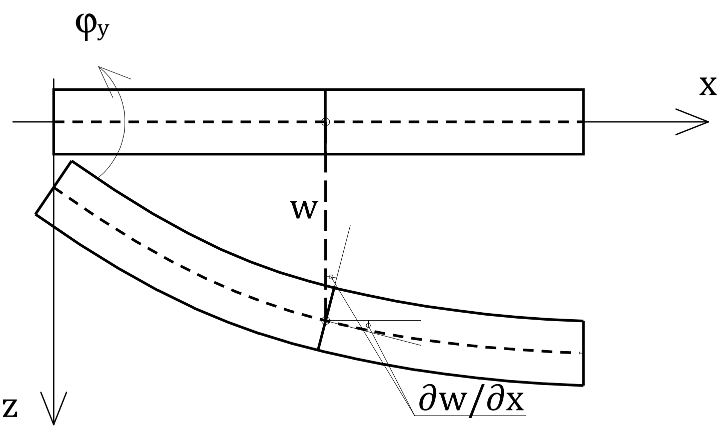


Figure 2: Displacement representation in the x-z plane

$$\varphi_y = -\frac{\partial w}{\partial x} \ ; \tag{2}$$

$$u(x,y,z) = \varphi_y(x,y) \cdot z = -\frac{\partial w(x,y)}{\partial x} \cdot z \ . \tag{3}$$

1.2.2 Displacements in the y-z plane

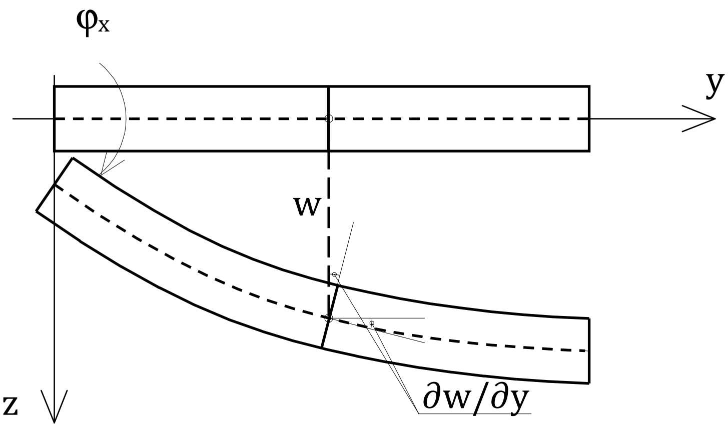


Figure 3: Displacement representation in the x-z plane

$$\varphi_x = \frac{\partial w}{\partial y} \;; \tag{4}$$

$$v(x,y,z) = -\varphi_x(x,y) \cdot z = -\frac{\partial w(x,y)}{\partial y} \cdot z \;. \tag{5}$$

Consequently, the following differential relationships completely define the displacement field of thin plates in bending:

$u = -z \cdot \frac{\partial w}{\partial x}$;	$v = -z \cdot \frac{\partial w}{\partial y}$.	(6)

1.3 Deformation field

The strain expressions can be easily derived by using their general definition depending on the displacement components within the framework of the Theory of Continuum Mechanics:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z \cdot \frac{\partial^2 w}{\partial x^2} \;; \tag{7}$$

$$\varepsilon_{yy} = \frac{\partial v}{\partial y} = -z \cdot \frac{\partial^2 w}{\partial y^2} \;; \tag{8}$$

$$\varepsilon_{zz} = \frac{\partial w}{\partial z} = 0 \;; \tag{9}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z \cdot \frac{\partial^2 w}{\partial x \partial y} \;; \tag{10}$$

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = -\frac{\partial w}{\partial x} + \frac{\partial w}{\partial x} = 0 \;; \tag{11}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = -\frac{\partial w}{\partial y} + \frac{\partial w}{\partial y} = 0 \;. \tag{12}$$

The above derivation of the complete strain field of the plate, points out other peculiarity of the thin plate model deriving directly by the initial kinematic assumptions:

- equation (9) confirms that the general chord of the plate remains undeformed in z-direction;
- equations (11) and (12) remark that the model of thin plates is based on the hypothesis that the mentioned normal chord remains normal to the deformed configuration of the midsurface.

1.4 Constitutive relationships

The constitutive laws relating the strains and the corresponding stress field within an elastic body can be placed in the following general form:

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} \;; \tag{13}$$

D being the so-called stiffness matrix. Depending on the mechanical properties of materials, various possible expressions can be considered for the matrix **D**.

The case of orthotropic materials (in two dimensions as considered for the particular hypotheses introduced in the case of plates) is among the simplest and most utilized ones. Equation (14) reports a reduced representation of the stiffness matrix **D** pointing out only the explicit expression of the terms directly involved in the derivation of the stress components depending on the non-zero strain components:

$$\mathbf{D} = \frac{1}{1-\upsilon_{xy}\upsilon_{yx}} \begin{bmatrix} E_x & E_x\upsilon_{yx} & 0 & 0 & 0 & 0 \\ E_y\upsilon_{xy} & E_y & 0 & 0 & 0 & 0 \\ 0 & 0 & D_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & G_{xy}\cdot\left(1-\upsilon_{xy}\upsilon_{yx}\right) & 0 & 0 \\ 0 & 0 & 0 & 0 & D_{55} & 0 \\ 0 & 0 & 0 & 0 & 0 & D_{66} \end{bmatrix} . \tag{14}$$

The matrix terms in equation (14) provide the relationships between the non-zero strain components derived in the previous paragraph and the corresponding stresses.

However, it is important to point out that, while σ_z can be approximately neglected as a result of the "thinness" hypothesis remarked at the beginning of the present notes, the two shear stresses τ_{xz} and τ_{yz} cannot be neglected for the sake of equilibrium, although the corresponding shear deformations are zero according to equations (11) and (12). Indeed, they only vanish as a result of the simplified kinematical hypotheses assumed for describing the plate behaviour in bending.

1.5 Stress field

The combined application of equations (13) and (14) points out the final definitions of three of the components of the stress field:

$$\sigma_{xx} = \frac{E_x}{1-\nu_{xy}\nu_{yx}}\cdot\left(\varepsilon_{xx}+\nu_{yx}\varepsilon_{yy}\right) ; \tag{15}$$

$$\sigma_{yy} = \frac{E_y}{1-\nu_{xy}\nu_{yx}}\cdot\left(\varepsilon_{yy}+\nu_{xy}\varepsilon_{xx}\right) ; \tag{16}$$

$$\tau_{xy} = G_{xy}\gamma_{xy} . \tag{17}$$

The other components can be derived by considering the equilibrium equation for continua in the absence of body forces:

$$\frac{\partial\sigma_{xx}}{\partial x}+\frac{\partial\tau_{xy}}{\partial y}+\frac{\partial\tau_{xz}}{\partial z}=0 , \tag{18}$$

$$\frac{\partial\tau_{yx}}{\partial x}+\frac{\partial\sigma_{yy}}{\partial y}+\frac{\partial\tau_{yz}}{\partial z}=0 . \tag{19}$$

Considering the former one, the following differential equation can be derived for τ_{xz}:

$$\frac{\partial \tau_{xz}}{\partial z} = -\frac{\partial \sigma_{xx}}{\partial x} - \frac{\partial \tau_{xy}}{\partial y} \quad , \tag{20}$$

$$\frac{\partial \tau_{yz}}{\partial z} = -\frac{\partial \tau_{yx}}{\partial x} - \frac{\partial \sigma_{yy}}{\partial y} \quad . \tag{21}$$

The above relationships should be integrated with respect to z and the final relationship for τ_{xz} and τ_{yz} throughout the plate thickness could be derived by imposing that they should be zero on the two faces (namely, for $z=\pm h/2$).

1.6 Generalized stresses

Starting from the kinematic assumptions for the plate body, simplified relationships have been derived for describing displacement, strain and stress fields.

In particular, equations (7), (8) and (9) suggest the definition of three "generalized" strain parameters related to the "curvature" of the plate along the two main directions:

$$\chi_y = -\frac{\partial^2 w}{\partial x^2} \quad , \tag{22}$$

$$\chi_x = -\frac{\partial^2 w}{\partial y^2} \quad , \tag{23}$$

$$\chi_{xy} = -\frac{\partial^2 w}{\partial x \partial y} \quad . \tag{24}$$

Generalized stresses can be defined starting for the point stresses determined in the previous paragraph with the aim of relating them to the above generalized strain parameters.

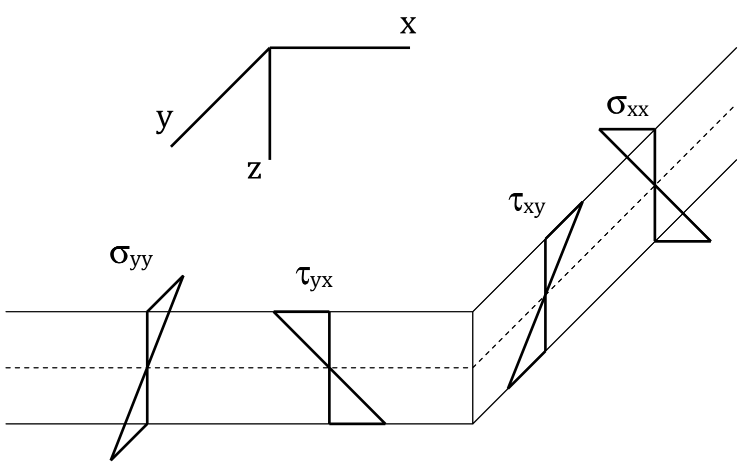


Figure 4: Point stresses on the positive face of a segmental plate element

In particular, the three stress components evaluated by equations (15), (16) and (17), emerging on the positive faces of the plate lateral boundary, have been represented in Figure 4. The following generalized stresses can be defined:

$$M_{xx} = \int_{-h/2}^{h/2} z \cdot \sigma_{xx} \cdot dz \ , \tag{25}$$

$$M_{yy} = \int_{-h/2}^{h/2} z \cdot \sigma_{yy} \cdot dz \ , \tag{26}$$

$$M_{xy} = -\int_{-h/2}^{h/2} z \cdot \tau_{xy} \cdot dz \ , \tag{27}$$

$$M_{yx} = \int_{-h/2}^{h/2} z \cdot \tau_{yx} \cdot dz \ . \tag{28}$$

It is worth to precise that in the previous equations the torques have been assumed positive according to the reference system represented in Figure 4.

Introducing equation (15) in (26) the following expression relating the bending moment M_{xx} to the above mentioned curvatures can be derived:

$$M_{xx} = -\left(D_x \cdot \frac{\partial^2 w}{\partial x^2} + D_1 \cdot \frac{\partial^2 w}{\partial y^2} \right) , \tag{29}$$

where

$$D_x = \frac{E_x h^3}{12 \cdot \left(1 - \nu_{xy}\nu_{yx}\right)} \quad \text{and} \quad D_1 = \frac{\nu_{yx} E_x h^3}{12 \cdot \left(1 - \nu_{xy}\nu_{yx}\right)} \ . \tag{30}$$

Following the same procedure, the following definition of the bending moment in M_{yy} can be derived:

$$M_{yy} = -\left(D_1 \cdot \frac{\partial^2 w}{\partial x^2} + D_y \cdot \frac{\partial^2 w}{\partial y^2} \right) , \tag{31}$$

with

$$D_y = \frac{E_y h^3}{12 \cdot \left(1 - \nu_{xy}\nu_{yx}\right)} \quad \text{and} \quad D_1 = \frac{\nu_{xy} E_y h^3}{12 \cdot \left(1 - \nu_{xy}\nu_{yx}\right)} = \frac{\nu_{yx} E_x h^3}{12 \cdot \left(1 - \nu_{xy}\nu_{yx}\right)} \ . \tag{32}$$

Finally, the torques can be also defined as functions of the conjugated curvature as follows:

$$M_{xy} = 2 D_{xy} \frac{\partial^2 w}{\partial x \partial y} \ , \tag{33}$$

$$M_{yx} = -2 D_{yx} \frac{\partial^2 w}{\partial x \partial y} \ , \tag{34}$$

where

$$D_{xy} = D_{yx} = G_{xy} \frac{h^3}{12} \quad . \qquad (35)$$

Furthermore, the shear forces acting on the x- and y-normal faces can be respectively derived as follows:

$$T_x = \int_{-h/2}^{h/2} \tau_{xz} dz \quad , \qquad (36)$$

$$T_y = \int_{-h/2}^{h/2} \tau_{yz} dz \quad . \qquad (37)$$

1.7 Equilibrium conditions

Before of deriving the relations between the above generalized stresses as a result of the equilibrium conditions, it is useful to remark the convention utilized so far for defining the positive direction of the generalized stresses:

- bending moments are assumed positive if they induce tension on the bottom face and tension of the top one;
- torques have been considered positive according to the positive directions of the axes;
- shear stresses are assumed positive if represented by a positive force on the positive face of the plate.

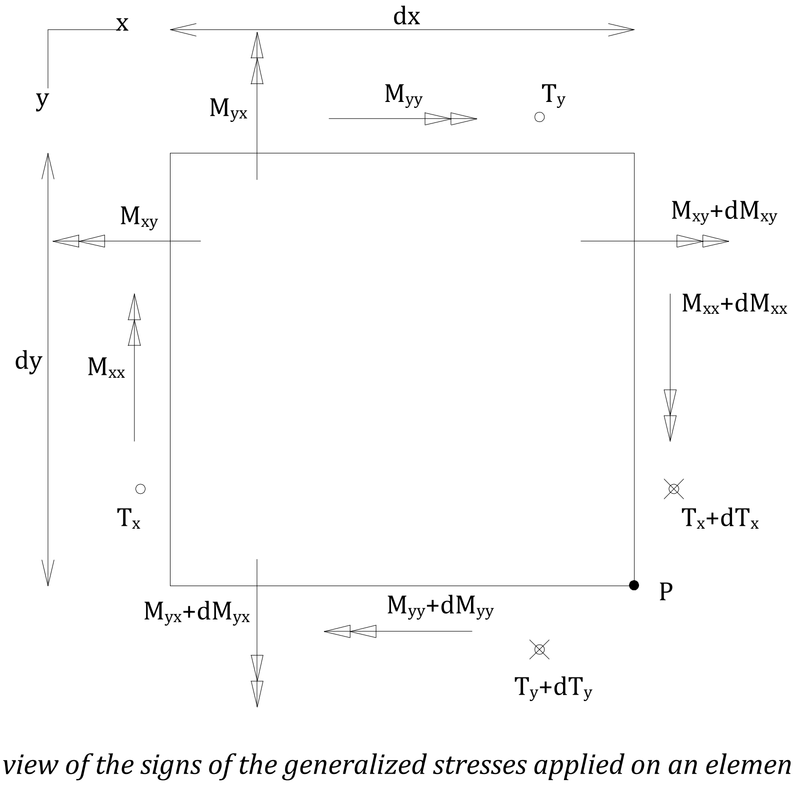


Figure 5: Plan view of the signs of the generalized stresses applied on an elementary part of the plate

Considering a transverse load p_0, ideally applied on the midsurface of the thin plate (practically, rather applied on one or both of their larger faces), a first equation derives by imposing the equilibrium condition in z direction:

$$\left(-T_x + T_x + \frac{\partial T_x}{\partial x}dx\right)dy + \left(-T_y + T_y + \frac{\partial T_y}{\partial y}dy\right)dx + p_0 dx dy = 0 \ , \tag{38}$$

and, after the due simplification:

$$\frac{\partial T_x}{\partial x} + \frac{\partial T_y}{\partial y} + p_0 = 0 \ . \tag{39}$$

The equilibrium condition can be also imposed in terms of rotation, i.e. around the point P; first of all the equilibrium of moments in y-direction can be considered deriving the following relationship:

$$\left(-M_{xx} + M_{xx} + \frac{\partial M_{xx}}{\partial x}dx\right)dy + \left(-M_{yx} + M_{yx} + \frac{\partial M_{yx}}{\partial y}dy\right)dx - T_x dy dx = 0 \ , \tag{40}$$

and the final differential equation:

$$T_x = \frac{\partial M_{xx}}{\partial x} + \frac{\partial M_{yx}}{\partial y} \ ., \tag{41}$$

Following a similar procedure, the rotation equilibrium around the point P can be considered also for moments in the x-direction:

$$\left(+M_{yy} - M_{yy} - \frac{\partial M_{yy}}{\partial y}dy\right)dx + \left(-M_{xy} + M_{xy} + \frac{\partial M_{xy}}{\partial x}dx\right)dy + T_y dy dx = 0 \ , \tag{42}$$

and the final differential equation can be derived by simplifying the above one and introducing M_{yx} *in lieu of* M_{xy} according to equations (33) and (34):

$$T_y = \frac{\partial M_{yx}}{\partial x} + \frac{\partial M_{yy}}{\partial y} \ . \tag{43}$$

1.8 Differential equation in terms of transverse displacement

Introducing equations (41) and (43) in (39), the following equation can be obtained:

$$\frac{\partial^2 M_{xx}}{\partial x^2} + 2 \cdot \frac{\partial M_{yx}}{\partial x} + \frac{\partial^2 M_{yy}}{\partial y^2} + p_0 = 0 \ , \tag{44}$$

and, introducing therein equation (29), (31) and (34), the following final equation in terms of the transverse displacement w(x,y) can be finally written:

$$D_x \frac{\partial^4 w}{\partial x^4} + 2H \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} = p_0 \quad , \tag{45}$$

where

$$H = D_1 + 2D_{xy} \quad . \tag{46}$$

2. Approximate Solutions

Several methods for solving the partial-differential equation (45) taking into account the boundary conditions deriving by either restraints or loads applied on the boundary of the plate have been developed. In the present section, the method based on double Fourier series will be explained and applied.

2.1 Solutions by double Fourier series

Approximate solutions of the problem can be easily found for rectangular and simply supported plates by considering the double Fourier series. In particular, every function f=f(x,y) of two variables x and y can be generally expressed as a double series of sine and cosine terms.

Since such a method can be utilized for approximating the function w=w(x,y) whose boundary value is zero as a result of the mentioned support conditions, only the sine terms can be utilized in the present case. Consequently, every function f=f(x,y) can be expressed as follows:

$$f(x,y) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} f_{ij} \cdot \sin\frac{i\pi x}{a} \cdot \sin\frac{j\pi y}{b} \ , \tag{47}$$

in which, as a result of the properties of the integral of sine functions, the following relationship can be stated for the coefficient f_{ij}:

$$f_{ij} = \frac{4}{ab} \cdot \int_0^a\int_0^b f(x,y) \cdot \sin\frac{i\pi x}{a} \cdot \sin\frac{j\pi y}{b} \cdot dxdy \ . \tag{48}$$

Consequently, if the load can be represented through the double Fourier series as follows:

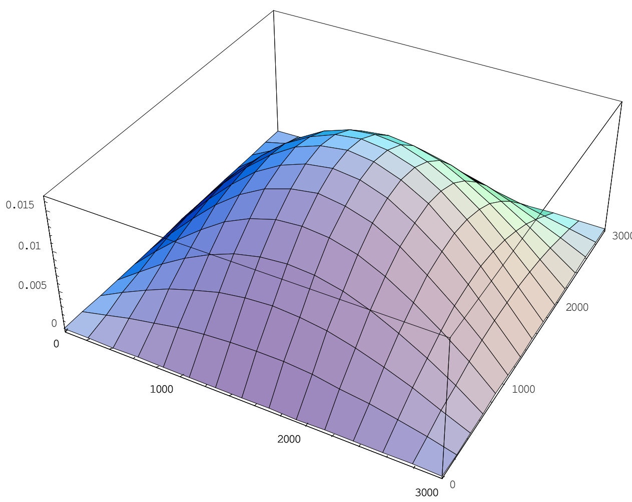
$$p(x,y) = \sum_{j=1}^{\infty}\sum_{i=1}^{\infty} p_{ij} \cdot \sin\frac{i\pi x}{a} \cdot \sin\frac{j\pi y}{b} \ , \tag{49}$$

with the following expression of the general terms of the series:

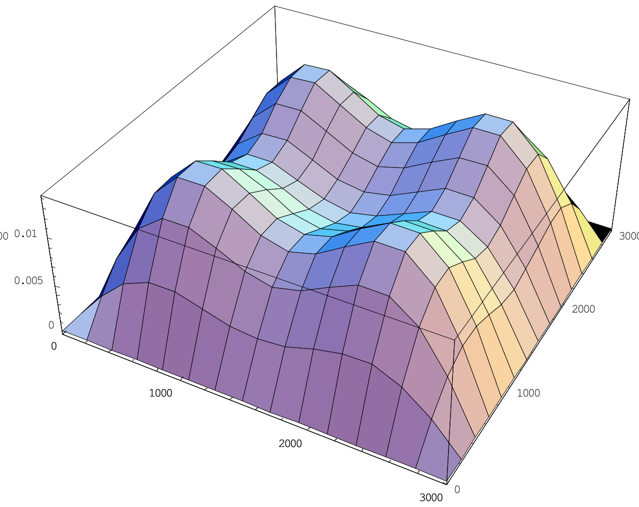
$$p_{ij} = \frac{4}{ab} \cdot \int_0^a\int_0^b p(x,y) \cdot \sin\frac{i\pi x}{a} \cdot \sin\frac{j\pi y}{b} \cdot dxdy \ . \tag{50}$$

the following relationship can be derived in the rather common case of uniformly distributed load $p(x,y)=p_0$:

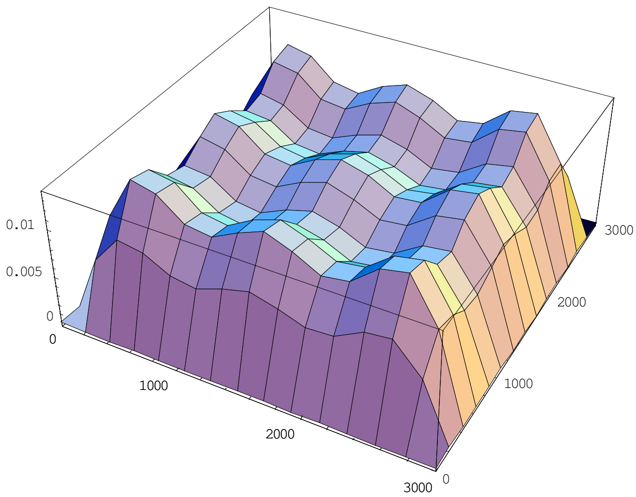
$$p_{ij} = \frac{4p_0}{ab} \cdot \left(\int_0^a \sin \frac{i\pi x}{a} dx \right) \cdot \left(\int_0^b \sin \frac{j\pi y}{b} dy \right) = \frac{16p_0}{i \cdot j \cdot \pi^2} \qquad i, j=1,3,5,7\dots . \tag{51}$$



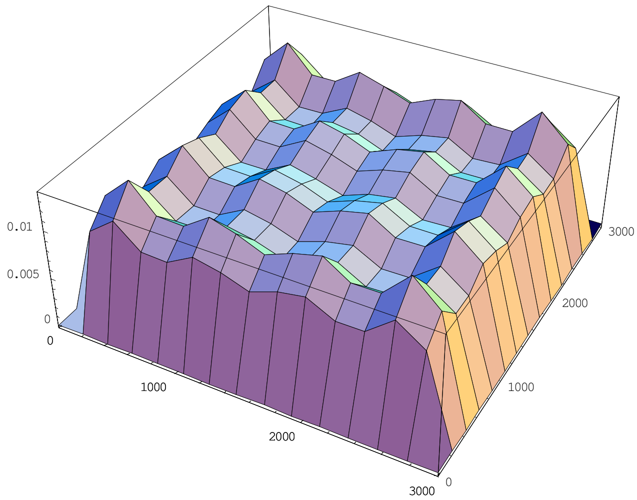
a) m=n=1



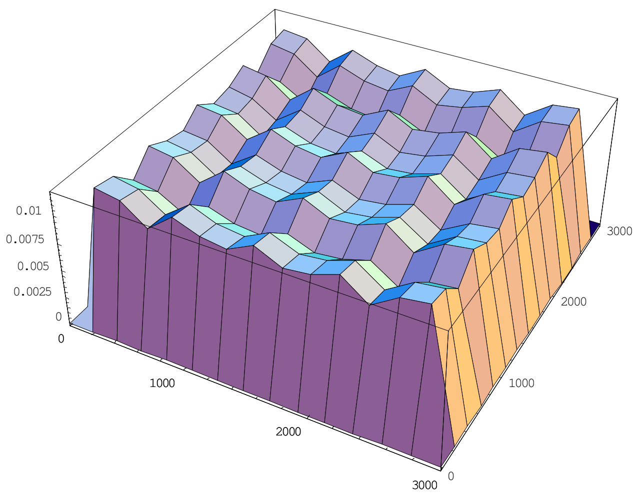
b) m=n=3



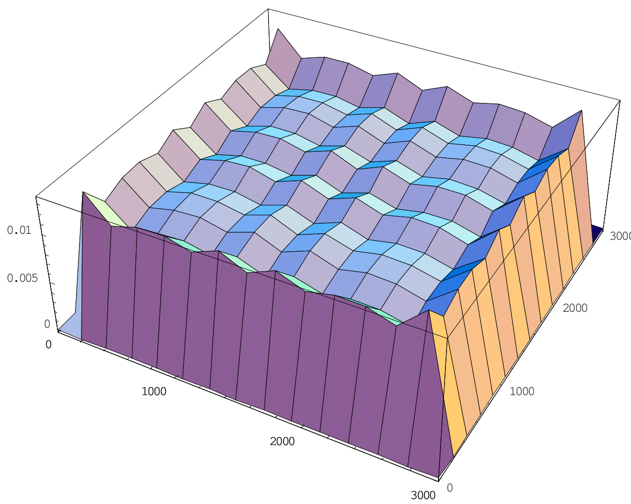
a) m=n=5



b) m=n=7



a) m=n=9



b) m=n=11

Figure 6: Progressive approximations of the load function – uniformly distributed load

Graphical examples on the quality of the approximation of a constant load by means of a richer and richer sine series are represented in Figure 6 plotting the series in equation (49) truncated at m and n as follows:

$$p(x,y)=\sum_{j=1}^{n}\sum_{i=1}^{m}p_{ij}\cdot\sin\frac{i\pi x}{a}\cdot\sin\frac{j\pi y}{b}\quad . \tag{52}$$

Representing a function through its truncated Fourier series conceptually consists in reducing the dimension of the structural problem at a dimension related to the values of m and n.

After expressing the displacement function w=w(x,y) in Fuorier series:

$$w(x,y)=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}w_{ij}\cdot\sin\frac{i\pi x}{a}\cdot\sin\frac{j\pi y}{b}\quad . \tag{53}$$

such expression can be introduced in into the equation (45) along with the expression of p based on equation (49) for deriving the coefficients w_{ij} depending by p_{ij}:

$$\begin{aligned}&\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}\left[D_x\left(\frac{i\pi}{a}\right)^4+2H\left(\frac{i\pi}{a}\right)^2\left(\frac{j\pi}{b}\right)^2+D_y\left(\frac{j\pi}{b}\right)^4\right]w_{ij}\cdot\sin\frac{i\pi x}{a}\cdot\sin\frac{j\pi y}{b}=\\&=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}p_{ij}\cdot\sin\frac{i\pi x}{a}\cdot\sin\frac{j\pi y}{b}\end{aligned}\quad , \tag{54}$$

and, finally:

$$w_{ij}=\frac{p_{ij}}{\pi^4\left[D_x\left(\frac{i}{a}\right)^4+2H\left(\frac{i}{a}\right)^2\left(\frac{j}{b}\right)^2+D_y\left(\frac{j}{b}\right)^4\right]}=\frac{\dfrac{16p_0}{\pi^6\cdot i\cdot j}}{D_x\left(\frac{i}{a}\right)^4+2H\left(\frac{i}{a}\right)^2\left(\frac{j}{b}\right)^2+D_y\left(\frac{j}{b}\right)^4} \tag{55}$$

i, j=1,3,5,7… .

Consequently, the complete expression of the displacement function can be placed in the following shape:

$$w(x,y)=\frac{16p_0}{\pi^6}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{1}{i\cdot j\cdot\left[D_x\left(\frac{i}{a}\right)^4+2H\left(\frac{i}{a}\right)^2\left(\frac{j}{b}\right)^2+D_y\left(\frac{j}{b}\right)^4\right]}\cdot\sin\frac{i\pi x}{a}\cdot\sin\frac{j\pi y}{b} \tag{56}$$

i, j=1,3,5,7… .

2.2 Application of the method to a RC plate supported on the boundary

The orthotropic plate model can be proficiently utilized for simulating the key aspects of the behaviour of various structural systems; those listed below are among the most common ones:

- reinforced concrete plates, whose anisotropy (even being often neglected in the practical applications) theoretically derives by the possibly different amount of steel reinforced placed in the two main directions;
- steel-concrete composite bridge decks, in which the two order of steel beams in the two directions of the bridge (namely, parallel and normal to the bridge axis) result in an orthotropic behaviour of the deck;
- laminated composite materials, made out of two materials (usually called “phases”), namely a *matrix*, usually made out epoxy-resin, and some *fibers*, possibly carbon-, glass- or aramid-based, directed in one or two (but even four or more) direction in plan.

The bending behaviour of the above materials as well as the one of other structural systems can be reproduced through the orthotropic plate model.

For instance, a possible application of that model can be proposed by considering a RC plate; the following calculations assume that no cracking occurs in concrete.

Since anisotropy basically derives by the possibly different amount of steel rebars in the two directions, we can assumed that both Young moduli and Poisson ratios in the two directions are those of concrete:

$$E_x = E_y = E_c \ , \qquad \nu_{xy} = \nu_{yx} = \nu_c \ . \tag{57}$$

The stiffness coefficients defined in equation (45) can be evaluated for the particular problem at hand by considering the homogenization procedure for all the relevant geometric properties of concrete; such procedure results by the two following hypotheses:

- linear behaviour of the two materials (namely, concrete and steel) described by their Young moduli E_c and E_s, respectively;
- no relative deformation between steel rebars and concrete.

Starting by those hypotheses, the equivalent moment of inertia of the plate section in the two direction can be easily derived once the value of the modular ratio $n_{eq}=E_s/E_c$ ha been defined. Consequently, the following expression of the stiffness coefficients involved in equation (45) are defined as follows

$$D_x^{eq} = \frac{E_c I_{c,x}^{eq}}{b\left(1-\upsilon_c^2\right)} = \frac{E_c}{b\left(1-\upsilon_c^2\right)} \cdot \left[I_{c,x} + \left(n_{eq}-1\right) \cdot I_{s,x} \right] \ , \tag{58}$$

$$D_y^{eq} = \frac{E_c I_{c,y}^{eq}}{a\left(1-\upsilon_c^2\right)} = \frac{E_c}{a\left(1-\upsilon_c^2\right)} \cdot \left[I_{c,y} + \left(n_{eq}-1\right) \cdot I_{s,y} \right] \ . \tag{59}$$

Since the following transformation can be considered in the definition of the coefficient D_l defined in (30)

$$D_l^{eq} = \frac{\nu_c E_x I_x^{eq}}{12 \cdot b \cdot \left(1-\nu_c^2\right)} = \frac{\nu_c E_y I_y^{eq}}{12 \cdot a \cdot \left(1-\nu_c^2\right)} = \nu_c D_x^{eq} = \nu_c D_y^{eq} \tag{60}$$

it can be written in the following form:

$$D_l^{eq} = \nu_c \sqrt{D_x^{eq} D_y^{eq}} \ , \tag{61}$$

and for similar reasons the following transformation can be considered for the coefficient D_{xy}:

$$D_{xy}^{eq} = \frac{G_c I_{c,x}^{eq}}{b} = \frac{G_c I_{c,y}^{eq}}{a} = \frac{E_c}{2 \cdot (1+\nu_c)} \frac{I_{c,x}^{eq}}{b} = \frac{E_c}{2 \cdot (1+\nu_c)} \frac{I_{c,y}^{eq}}{a} \ , \tag{62}$$

and finally

$$D_{xy}^{eq} = \frac{E_c \cdot (1-\nu_c)}{2 \cdot \left(1-\nu_c^2\right)} \frac{I_{c,x}^{eq}}{b} = \frac{E_c \cdot (1-\nu_c)}{2 \cdot \left(1-\nu_c^2\right)} \frac{I_{c,y}^{eq}}{a} = \frac{(1-\nu_c)}{2} \cdot \sqrt{D_x^{eq} D_y^{eq}} \ . \tag{63}$$

Finally, since

$$H_{eq} = D_l^{eq} + 2 D_{xy}^{eq} = \sqrt{D_x^{eq} D_y^{eq}} \ , \tag{64}$$

and equation (45) can be finally placed in the following shape for the problem at hand:

$$D_x^{eq} \frac{\partial^4 w}{\partial x^4} + \sqrt{D_x^{eq} D_y^{eq}} \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y^{eq} \frac{\partial^4 w}{\partial y^4} = p_0 \ , \tag{65}$$

or, equivalently,

$$\frac{\partial^4 w}{\partial x^4} + \sqrt{\frac{D_y^{eq}}{D_x^{eq}}} \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{D_y^{eq}}{D_x^{eq}} \frac{\partial^4 w}{\partial y^4} = \frac{p_0}{D_x^{eq}} \ . \tag{66}$$

Introducing the following scale change in y-direction,

$$y_1 = \sqrt[4]{\frac{D_x^{eq}}{D_y^{eq}}} \cdot y \ , \tag{67}$$

the following changes have to be introduced in the partial derivatives:

$$\frac{\partial^2 w}{\partial y^2} = \sqrt{\frac{D_x^{eq}}{D_y^{eq}}} \frac{\partial^2 w}{\partial y_1^2} \ , \qquad \frac{\partial^4 w}{\partial y^4} = \frac{D_x^{eq}}{D_y^{eq}} \frac{\partial^4 w}{\partial y_1^4} \ , \tag{68}$$

and, finally, the equation (66) can be transformed to obtain a shape formally equivalent to the one derived for the case of isotropic material:

$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y_1^2} + \frac{\partial^4 w}{\partial y_1^4} = \frac{p_0}{D_x^{eq}} \ . \tag{69}$$

Finally, the general solution in double Fourier series described in equation (56) can be simplified as follows as a result of the formal transformation of the above equation:

$$w(x,y_1)=\frac{16p_0}{\pi^6}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\sin\frac{i\pi x}{a}\cdot\sin\frac{j\pi y_1}{b}}{i\cdot j\cdot D_x^{eq}\left[\left(\frac{i}{a}\right)^2+\left(\frac{j}{b}\right)^2\right]^2}\qquad \text{i, j=1,3,5,7… .} \tag{70}$$

The maximum displacement of the plate is achieved in the mid point of its plan and consequently for a point of coordinate (x,y)=(a/2,b/2):

$$w_{max}=\frac{16p_0}{\pi^6}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{\sin\frac{i\pi}{2}\cdot\sin\frac{j\pi}{2}}{i\cdot j\cdot D_x^{eq}\left[\left(\frac{i}{a}\right)^2+\left(\frac{j}{b}\right)^2\right]^2}\qquad \text{i, j=1,3,5,7… .} \tag{71}$$

The following algebraic transformation can be finally introduced in the above expression for simplifying its numerical application:

$$w_{max}=\frac{16p_0a^4}{\pi^6\cdot D_x^{eq}}\sum_{i=1}^{\infty}\sum_{j=1}^{\infty}\frac{(-1)^{\frac{m+n}{2}-1}}{i\cdot j\left[(i)^2+\left(\frac{a}{b}j\right)^2\right]^2}\qquad \text{i, j=1,3,5,7… .} \tag{72}$$

Finally, an well approximate solution can be obtained by truncating the above series at the third terms in both i and j:

$$w_{max}\approx\frac{16p_0a^4}{\pi^6\cdot D_x^{eq}}\left\{\frac{1}{\left[1+\left(\frac{a}{b}\right)^2\right]^2}-\frac{1}{3\cdot\left[9+\left(\frac{a}{b}\right)^2\right]^2}-\frac{1}{3\cdot\left[1+9\cdot\left(\frac{a}{b}\right)^2\right]^2}+\frac{1}{729\cdot\left[1+1\cdot\left(\frac{a}{b}\right)^2\right]^2}\right\}. \tag{73}$$

and in the case of square plate:

$$w_{max}\approx\frac{16p_0a^4}{\pi^6\cdot D_x^{eq}}\cdot\left\{\frac{1}{4}-\frac{1}{300}-\frac{1}{300}+\frac{1}{2916}\right\}\approx 0{,}00406\cdot\frac{p_0a^4}{D_x^{eq}}\ . \tag{74}$$

3. References

[1]. Timoshenko S.P., Woinkowsky-Krieger S.: *Theory of Plates and Shells*, McGraw-Hill Book Company, NY. 2nd edition 1959, reissued 1987;

[2]. Felippa C.: *Kirchhoff Plates: Field equations*, in "Advanced Finite Element Methods", (http://www.colorado.edu/engineering/CAS/courses.d/AFEM.d/).