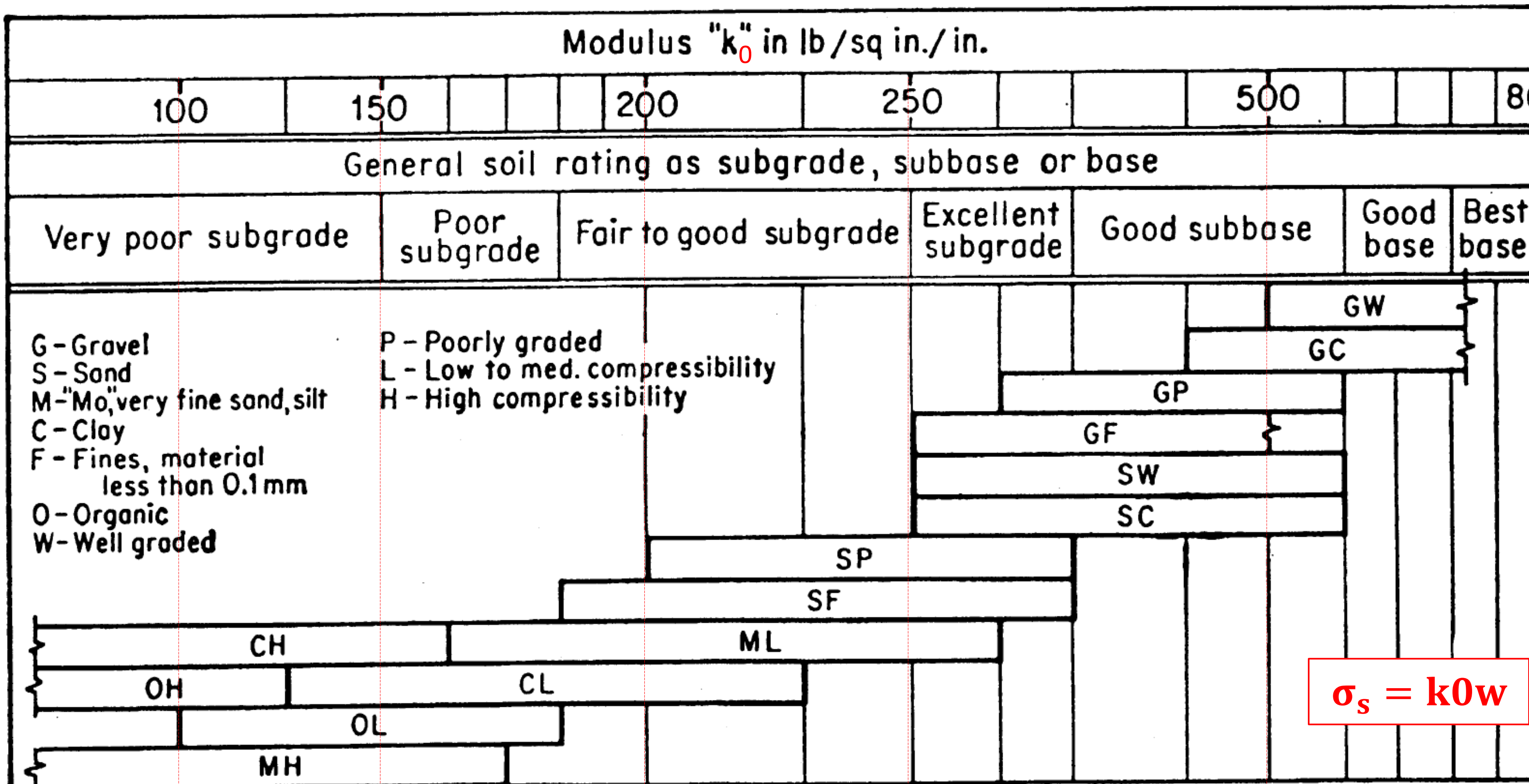
Plates on elastic foundation

Circular elastic plate, axial-symmetric load, Winkler soil

(after *Timoshenko & Woinowsky-Krieger* (1959) - Chapter 8)

Introduction

Winkler model: modulus of subgrade



$$\sigma_s = k0w$$

N/mm^3	0.0271	0.0407	0.0543	0.0679	0.136
kg$_f$/cm^3	2.76	4.15	5.53	6.92	13.83

Introduction

Circular elastic (thin) plates

The *Kirchhoff–Love theory* of plates is a two-dimensional mathematical model that is used to determine the stresses and deformations in thin plates subjected to forces and moments.

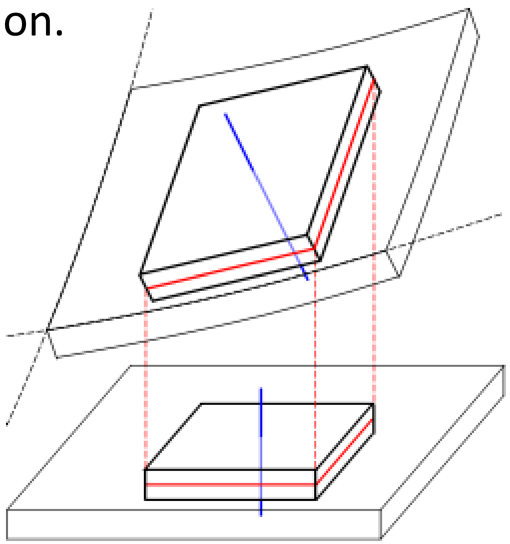
The following *kinematic assumptions* that are made in this theory:

- straight lines normal to the mid-surface remain straight after deformation
- straight lines normal to the mid-surface remain normal to the mid-surface after deformation
- the thickness of the plate does not change during a deformation.

(after *Wikipedia*)

Germain-Lagrange equation (cartesian coordinates)

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$



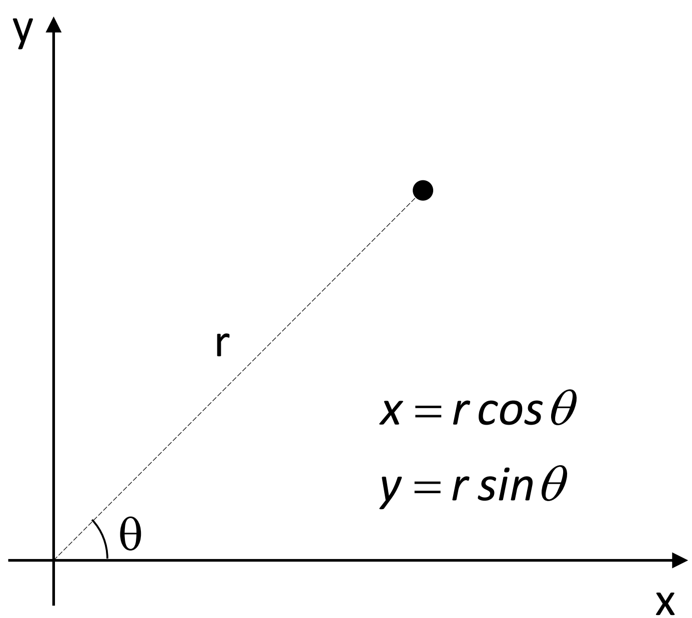
Introduction

Circular elastic (thin) plates

Germain-Lagrange equation (cartesian coordinates)

$$\Delta_2 w = \Delta\Delta w = \frac{q}{D} \qquad \Delta(\bullet) = \frac{\partial^2(\bullet)}{\partial x^2} + \frac{\partial^2(\bullet)}{\partial y^2}$$

Conversion to polar (cylindrical) coordinates



$$x = r\cos\theta$$

$$y = r\sin\theta$$

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial r}\cdot\cos\theta - \frac{1}{r}\cdot\frac{\partial w}{\partial\theta}\cdot\sin\theta$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial r}\cdot\sin\theta + \frac{1}{r}\cdot\frac{\partial w}{\partial\theta}\cdot\cos\theta$$

$$\Delta(\bullet) = \frac{\partial^2(\bullet)}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial(\bullet)}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2(\bullet)}{\partial^2\theta}$$

Introduction

Circular elastic (thin) plates under axial-symmetric load

Germain-Lagrange equation (polar coordinates)

$$\Delta_2 w = \Delta\Delta w = \frac{q}{D} \qquad \Delta(\bullet) = \frac{\partial^2(\bullet)}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial(\bullet)}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2(\bullet)}{\partial^2\theta}$$

$$\Delta_2 w = \left(\frac{\partial^2(\bullet)}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial(\bullet)}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2(\bullet)}{\partial^2\theta}\right)\left(\frac{\partial^2 w}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial w}{\partial r} + \frac{1}{r^2}\cdot\frac{\partial^2 w}{\partial^2\theta}\right) = \frac{q}{D}$$

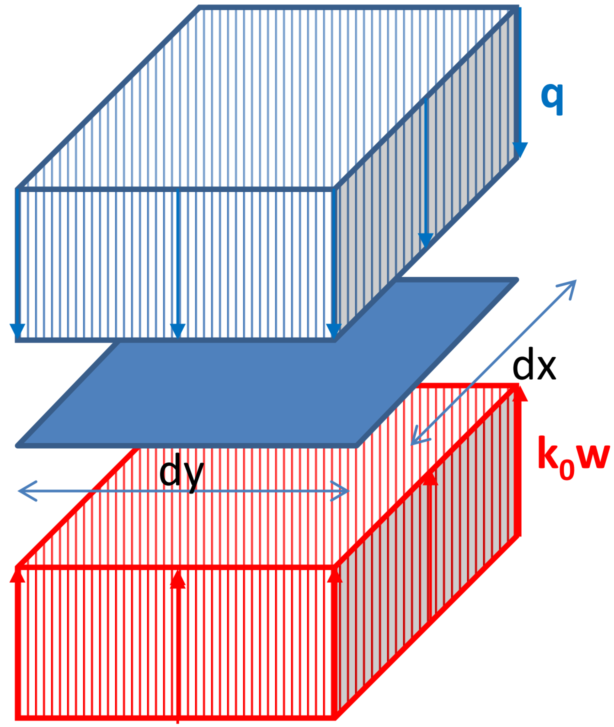
$$\boxed{\left(\frac{d^2}{dr^2} + \frac{1}{r}\cdot\frac{d}{dr}\right)\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\cdot\frac{dw}{dr}\right) = \frac{q}{D}}$$

Introduction

Circular elastic (thin) plates on Winkler soil under axial-symmetric load

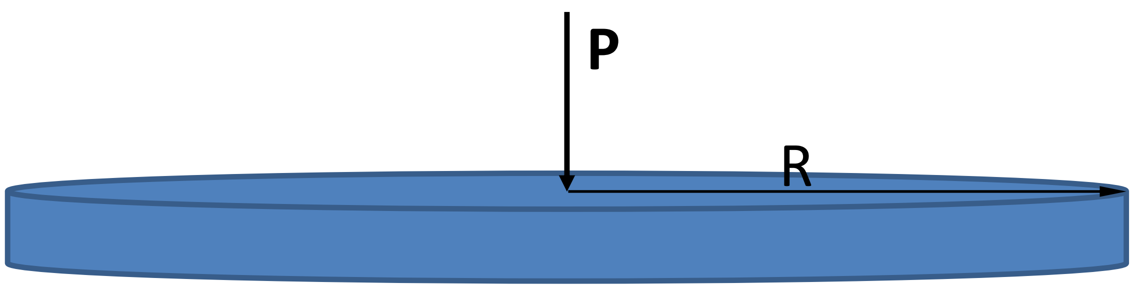
$$\left(\frac{d^2}{dr^2}+\frac{1}{r}\cdot\frac{d}{dr}\right)\left(\frac{d^2w}{dr^2}+\frac{1}{r}\cdot\frac{dw}{dr}\right)=\frac{q-k_0w}{D}$$

$$\frac{d^4w}{dr^4}+\frac{2}{r}\frac{d^3w}{dr^3}-\frac{1}{r^2}\frac{d^2w}{dr^2}+\frac{1}{r^3}\frac{dw}{dr}=\frac{q-k_0w}{D}$$



Since an uniformly distributed load q applied on an elastic plates (or elastic beams) on Winkler soils with uniform k_0 does not determine any stress within the plate, the sole effect of a point load P applied in the centre of the plate is considered hereafter.

q=0



Approximate solution

Conversion to a dimensionless form

- Characteristic length:

$$\frac{k_0}{D} = \frac{1}{l^4} \qquad \frac{[F][L]^{-3}}{[F][L]} = \frac{1}{[L]^4}$$

- Dimensionless quantities:

Dimensionless displacement $z = \frac{w}{l}$

Dimensionless radius $x = \frac{r}{l}$

Dimensionless derivatives:

$$\frac{dw}{dr} = \frac{l \cdot dz}{l \cdot dx} = \frac{dz}{dx}$$

$$\frac{d^2 w}{dr^2} = \frac{d}{dr}\frac{dw}{dr} = \frac{d}{l \cdot dx}\frac{dz}{dx} = \frac{1}{l}\frac{d^2 z}{dx^2}$$

$$\frac{d^i w}{dr^i} = \frac{d}{dr}\frac{dw}{dr} = \frac{1}{l^{i-1}} \cdot \frac{d^i w}{dx^i}$$

$$\left(\frac{d^2}{dr^2} + \frac{1}{r}\cdot\frac{d}{dr}\right)\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\cdot\frac{dw}{dr}\right) + \frac{w}{l^4} = 0$$

$$\frac{1}{l^3}\cdot\left(\frac{d^2}{dx^2} + \frac{1}{x}\cdot\frac{d}{dx}\right)\left(\frac{d^2 z}{dx^2} + \frac{1}{x}\cdot\frac{dz}{dx}\right) + \frac{z}{l^3} = 0$$

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\cdot\frac{d}{dx}\right)\left(\frac{d^2 z}{dx^2} + \frac{1}{x}\cdot\frac{dz}{dx}\right) + z = 0$$

$$\frac{d^4 z}{dx^4} + \frac{2}{x}\frac{d^3 z}{dx^3} - \frac{1}{x^2}\frac{d^2 z}{dx^2} + \frac{1}{x^3}\frac{dz}{dx} + z = 0$$

Approximate solution

General solution

The proposed mathematical transformations led to the following *homogeneous 4th-order linear differential equation with variable coefficients*:

$$\frac{d^4z}{dx^4}+\frac{2}{x^3}\frac{d^3z}{dx^3}-\frac{1}{x^2}\frac{d^2z}{dx^2}+\frac{1}{x^3}\frac{dz}{dx}+z=0$$

Therefore, in principle the *general solution* of this differential equation might be written as follows:

$$z=A_1\cdot X_1(x)+A_2\cdot X_2(x)+A_3\cdot X_3(x)+A_4\cdot X_4(x)$$

where X_i are four independent solutions of the differential equation under consideration and A_i are four integration constants depending on the actual boundary conditions.

Note:

$$\left(\frac{d^2}{dx^2}+\frac{1}{x}\cdot\frac{d}{dx}\right)\left(\frac{d^2X_i}{dx^2}+\frac{1}{x}\cdot\frac{dX_i}{dx}\right)+X_i=0 \qquad i=1..4$$

Approximate solution

Approximation by power series

Since X_i are unknown (as the differential equation under consideration includes *variable coefficients*), an approximation based on power series may be firstly assumed. Hence, the general expression of Xi can be taken as follows:

$$X_i = \sum_{n=0}^{N_T} a_n x^n = a_0 + a_1 x^1 + a_2 x^2 + \ldots + a_n x^n + \ldots + a_{N_T} x^{N_T}$$

where N_T is the highest monomial in the series.

Since X_i should be a solution of the general differential equation, the 2[nd] order laplacian of each term $a_n x^n$ should find a similar monomial such that:

$$\Delta\Delta X_i + X_i = 0$$

$$\Delta\Delta a_n x^n = \left(\frac{d^2}{dx^2} + \frac{1}{x}\cdot\frac{d}{dx}\right)\left(\frac{d^2 a_n x^n}{dx^2} + \frac{1}{x}\cdot\frac{d a_n x^n}{dx}\right) \Rightarrow n^2 (n-2)^2 a_n x^{n-4} + a_{n-4} x^{n-4} = 0$$

$$\frac{d^2 a_n x^n}{dx^2} + \frac{1}{x}\cdot\frac{d a_n x^n}{dx} = n(n-1) a_n x^{n-2} + n a_n x^{n-2} = n^2 a_n x^{n-2}$$

$$\left(\frac{d^2}{dx^2} + \frac{1}{x}\cdot\frac{d}{dx}\right) n^2 a_n x^{n-2} = n^2 (n-2)(n-3) a_n x^{n-4} + n^2 (n-2) a_n x^{n-4} = n^2 (n-2)^2 a_n x^{n-4}$$

$$a_n = -\frac{a_{n-4}}{n^2 (n-2)^2}$$

Recursive definition of a_n

Approximate solution

Approximation by power series: first independed solution X_1

$$a_n = -\frac{a_{n-4}}{n^2\left(n-2\right)^2}$$

$$z = A_1 \cdot X_1\left(x\right) + A_2 \cdot X_2\left(x\right) + A_3 \cdot X_3\left(x\right) + A_4 \cdot X_4\left(x\right)$$

Based on the recursive definition of the an coefficients, a first independent solution X_1 can be "built up" by assuming a_0 as the first nonzero coefficient:

$$n = 0: \qquad a_0 = 1 \qquad\qquad a_1 = a_2 = a_3 = 0$$

$$n = 4: \qquad a_4 = -\frac{a_0}{4^2\left(4-2\right)^2} = -\frac{1}{4^2 \cdot 2^2}$$

$$n = 8: \qquad a_8 = -\frac{a_4}{8^2\left(8-2\right)^2} = \frac{1}{8^2 \cdot 6^2 \cdot 4^2 \cdot 2^2}$$

$$n = 12: \qquad a_{12} = -\frac{a_8}{12^2\left(12-2\right)^2} = \frac{1}{12^2 \cdot 10^2 \cdot 8^2 \cdot 6^2 \cdot 4^2 \cdot 2^2}$$

$$X_1\left(x\right) = 1 - \frac{1}{64}x^4 + \frac{1}{147456}x^8 - \frac{1}{2123366400}x^{12} \dots$$

Approximate solution

Approximation by power series: second independed solution X_2

$$a_n = -\frac{a_{n-4}}{n^2 (n-2)^2}$$

Before proceeding with "constructing" the second independent solution X_2 of z, it should be noted that the assumption of $a_1=1$ is not admissible.

In fact, if one assumes $a_1=1$ (and, at the same time, $a_0=a_2=a_3=0$) the resulting solution X_2 of z would have a first term x:

$$X_2 = x + \ldots.$$

which implies a nonzero first derivative of X_2 (and, hence, of the general solution z) which cannot be accepted because of the fact that rotations ($\varphi=dX_2/dx=1+\ldots$) should be zero at x=0.

Therefore, the second solution X_2 should be built by assuming $a_2=1$ and, at the same time, $a_0=a_1=a_3=0$:

$$X_2 = x^2 + \ldots.$$

this leads to an expression of X_2 whose first nonzero monomial is of 2nd order and, hence, the second derivative of X_2 is non zero ($\chi=d^2X_2/dx_2=1+\ldots$)): this is acceptable because the curvature is nonzero at the plate centre (x=0).

Approximate solution

Approximation by power series: second independed solution X_2

$$a_n = -\frac{a_{n-4}}{n^2\left(n-2\right)^2}$$

Based on the recursive definition of the an coefficients, a first independent solution X_2 can be "built up" by assuming a_2 as the first nonzero coefficient:

$n=2:\quad a_2 = 2 \qquad a_0 = a_1 = a_3 = 0$

$$n=6:\quad a_6 = -\frac{a_2}{6^2\left(6-2\right)^2} = -\frac{1}{6^2\cdot 4^2}$$

$$n=10:\quad a_{10} = -\frac{a_6}{10^2\left(10-2\right)^2} = \frac{1}{10^2\cdot 8^2\cdot 6^2\cdot 4^2}$$

$$n=14:\quad a_{14} = -\frac{a_{10}}{14^2\left(14-2\right)^2} = \frac{1}{14^2\cdot 12^2\cdot 10^2\cdot 8^2\cdot 6^2\cdot 4^2}$$

$$X_2\left(x\right) = x^2 - \frac{1}{576}x^6 + \frac{1}{3686400}x^{10} - \frac{1}{104044953600}x^{14}\ldots$$

Approximate solution

Approximation by power series: third independed solution X_3

$$a_n = -\frac{a_{n-4}}{n^2 (n-2)^2}$$

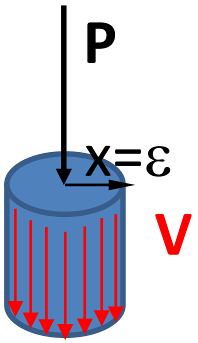
The assumption a_3=1 (and, at the same time, a_0=a_1=a_2=0) is not even admissible, as it would lead to the a function having the following expression:

$$X_3 = x^3 + ...$$

This is not admissible because it would lead to a finite value of shear for x=0:

$$V \propto \frac{d}{dx}\left(\frac{d^2 X_3}{dx^2} + \frac{1}{x}\frac{dX_3}{dx}\right) = 2 + ...$$

whereas shear should be divergent (it should tend to infinity) for $x \to 0$, as a finite force P should be in equilibrium with shear stresses on an infinitesimal circumference of radius ε.



Approximate solution

Approximation by power series: third independed solution X_3

Since,
on the one hand, shear V $\rightarrow\infty$ for x$\rightarrow$0,
and
on the other one hand, both X_1 and X_2 would lead to a finite value of shear for x=0,
the solutions X_3 and X_4 should be capable to reproduce the condition V $\rightarrow\infty$ for x$\rightarrow$0.

Therefore, a different solution, including the term log x (which diverges for x$\rightarrow$0 along with all its derivatives) ad an new unknown function F_3, is preliminarily defined:

$$X_3 = X_1 \cdot \log x + F_3$$

The function X_3 should be, in turn, a solution of the general differential equation:

$$\Delta\Delta X_i + X_i = 0 \quad \Longleftrightarrow \quad \left(\frac{d^2}{dx^2} + \frac{1}{x}\cdot\frac{d}{dx}\right)\left(\frac{d^2 X_i}{dx^2} + \frac{1}{x}\cdot\frac{dX_i}{dx}\right) + X_i = 0$$

First of all, the 2nd laplacian of X_3 is determined

$$\Delta\Delta X_3 = \frac{4}{x}\frac{d^3 X_1}{dx^3} + \log x \cdot \Delta\Delta X_1 + \Delta\Delta F_3$$

Approximate solution

Approximation by power series: third independed solution X_3

Then:

$$\Delta\Delta X_3 + X_3 = 0$$

The following condition has to be met:

$$\Delta\Delta X_3 + X_3 = \frac{4}{x}\frac{d^3 X_1}{dx^3} + \log x \cdot \Delta\Delta X_1 + \Delta\Delta F_3 + X_1 \log x + F_3 = 0$$

and, after collecting the term log x,

$$\frac{4}{x}\frac{d^3 X_1}{dx^3} + \log x \cdot \left(\Delta\Delta X_1 + X_1\right) + \Delta\Delta F_3 + F_3 = 0$$

the following relationship is derived for F_3:

$$\Delta\Delta F_3 + F_3 = -\frac{4}{x}\frac{d^3 X_1}{dx^3}$$

$$\Delta\Delta F_3 + F_3 = -4 \cdot \left(-\frac{2 \cdot 3 \cdot 4}{64} + \frac{6 \cdot 7 \cdot 8}{147456} x^4 - \frac{10 \cdot 11 \cdot 12}{2123366400} x^8 + \ldots \right)$$

Approximate solution

Approximation by power series: third independed solution X_3

$$\Delta\Delta F_3 + F_3 = -4\cdot\left(-\frac{2\cdot 3\cdot 4}{4^2\cdot 2^2} + \frac{6\cdot 7\cdot 8}{8^2\cdot 6^2\cdot 4^2\cdot 2^2}x^4 - \frac{10\cdot 11\cdot 12}{12^2\cdot 10^2\cdot 8^2\cdot 6^2\cdot 4^2\cdot 2^2}x^8 + \ldots\right)$$

$$\Delta\Delta F_3 + F_3 = 4\cdot\sum_{n=4}^{n_T}(-1)^{\frac{n}{4}-1}\cdot\frac{n\cdot(n-1)\cdot(n-2)}{n^2\cdot(n-2)^2\cdot\ldots\cdot 2^2}x^{n-4}$$

The function F_3 may also be approximated by the following power series:

$$F_3 = b_4x^4 + b_8x^8 + b_{12}x^{12} + \ldots$$

whose 2[nd] laplacian can be expressed as follows:

$$\Delta\Delta b_n x^n = n^2\left(n-2\right)^2 b_n x^{n-4}$$

$$n^2\left(n-2\right)^2 b_n x^{n-4} + b_{n-4}x^{n-4} = (-1)^{\frac{n}{4}-1}\cdot\frac{n\cdot(n-1)\cdot(n-2)}{n^2\cdot(n-2)^2\cdot\ldots\cdot 2^2}\cdot x^{n-4} \qquad \forall n$$

$$b_n = \frac{1}{n^2\left(n-2\right)^2}\left[-b_{n-4} + (-1)^{\frac{n}{4}-1}\cdot 4\cdot\frac{n\cdot(n-1)\cdot(n-2)}{n^2\cdot(n-2)^2\cdot\ldots\cdot 2^2}\right]$$

Approximate solution

Approximation by power series: third independed solution X_3

$$b_n = \frac{1}{n^2 (n-2)^2} \left[-b_{n-4} + (-1)^{\frac{n}{4}-1} \cdot 4 \cdot \frac{n \cdot (n-1) \cdot (n-2)}{n^2 \cdot (n-2)^2 \cdot .. \cdot 2^2} \right]$$

In[36]:= **ConstsSol3 = Solve[SimEq3 == 0, Constants3] // Flatten**

Out[36]= $\left\{ b4 \to \frac{3}{128},\ b8 \to -\frac{25}{1769472},\ b12 \to \frac{49}{42467328000},\ b16 \to -\frac{761}{2983176909619200},\right.$
$b20 \to \frac{7381}{34795775473798348800000},\ b24 \to -\frac{86021}{10670556016656138759045120000 0},$
$b28 \to \frac{1171733}{7351791147910933157295899541504 00000},\ b32 \to -\frac{2436559}{135508214438294319955278020349001728 0000000},$
$\left. b36 \to \frac{14274301}{1150419209821067509528805419348187406139392 0000000},\ b40 \to -\frac{11167027}{20200256922016517242516675510551596591897829703680000000} \right\}$

In[37]:= **X3[x] = Expand[FullSimplify[X3[x] /. ConstsSol3]]**

Out[37]= $\frac{3 x^4}{128} - \frac{25 x^8}{1769472} + \frac{49 x^{12}}{42467328000} - \frac{761 x^{16}}{29831769096192000} + \frac{7381 x^{20}}{34795775473798348800000} - \frac{86021 x^{24}}{106705560166561387590451200000} +$
$\frac{1171733 x^{28}}{735179114791093315729589954150400000} - \frac{2436559 x^{32}}{1355082144382943199552780203490017280000000} +$
$\frac{14274301 x^{36}}{11504192098210675095288054193481874061393920000000} - \frac{11167027 x^{40}}{20200256922016517242516675510551596591897829703680000000} +$
$\text{Log}[x] - \frac{1}{64} x^4 \text{Log}[x] + \frac{x^8 \text{Log}[x]}{147456} - \frac{x^{12} \text{Log}[x]}{2123366400} + \frac{x^{16} \text{Log}[x]}{106542032486400} - \frac{x^{20} \text{Log}[x]}{13807847410237440000} +$
$\frac{x^{24} \text{Log}[x]}{3849406932415634472960000} - \frac{x^{28} \text{Log}[x]}{2040124083669367620517232640000} + \frac{x^{32} \text{Log}[x]}{1880178355509689199068681601024000000} -$
$\frac{x^{36} \text{Log}[x]}{2816838087944084125503921126295732224000000} + \frac{x^{40} \text{Log}[x]}{6508022718386011963564259370193659730329600000000}$

Approximate solution

Approximation by power series: fourth independed solution X_4

A similar procedure leads to defining the fourth independent solution:

$$X_4 = X_2 \cdot \log x + F_4$$

and, similar considerations, can be done in defining F_4 for X_4 is a solution of the general differential equation.

$$F_4 = c_6 x^6 + c_{10} x^{10} + c_{14} x^{14} + \ldots$$

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In[49]:= ConstsSol4 = Solve[SimEq4 == 0, Constants4] // Flatten
```

Out[49]=

$$\left\{c6 \to \frac{5}{3456},\ c10 \to -\frac{77}{221184000},\ c14 \to \frac{223}{14566293504000},\ c18 \to -\frac{4609}{21747359671123968000},\right.$$
$$c22 \to \frac{55991}{46313177155625602252800000},\ c26 \to -\frac{785633}{234432115685935368536221286400000},$$
$$c30 \to \frac{835397}{1654153008279959960391577396838400000000},\ c34 \to -\frac{29889983}{665751857535339993940280913974645489664000000 0},$$
$$\left.c38 \to \frac{197698279}{789072536016270204785807637130921741871008972800000000},\ c42 \to -\frac{2736977}{2969437767536428034649951300051084699008980966440960000000}\right\}$$

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In[50]:= X4[x] = Expand[FullSimplify[X4[x] /. ConstsSol4]]
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Out[50]=

$$\frac{5 x^6}{3456} - \frac{77 x^{10}}{221184000} + \frac{223 x^{14}}{14566293504000} - \frac{4609 x^{18}}{21747359671123968000} + \frac{55991 x^{22}}{46313177155625602252800000} - \frac{785633 x^{26}}{234432115685935368536221286400000} +$$
$$\frac{835397 x^{30}}{1654153008279959960391577396838400000000} - \frac{29889983 x^{34}}{6657518575353399939402809139746454896640000000} +$$
$$\frac{197698279 x^{38}}{789072536016270204785807637130921741871008972800000000} - \frac{2736977 x^{42}}{2969437767536428034649951300051084699008980966440960000000} +$$
$$x^2 \operatorname{Log}[x] - \frac{1}{576} x^6 \operatorname{Log}[x] + \frac{x^{10} \operatorname{Log}[x]}{3686400} - \frac{x^{14} \operatorname{Log}[x]}{104044953600} + \frac{x^{18} \operatorname{Log}[x]}{8629904631398400} - \frac{x^{22} \operatorname{Log}[x]}{1670749536638730240000} +$$
$$\frac{x^{26} \operatorname{Log}[x]}{650549771578242225930240000} - \frac{x^{30} \operatorname{Log}[x]}{459027918825607714616377344000000} + \frac{x^{34} \operatorname{Log}[x]}{543371544742300178530848982695936000000} -$$
$$\frac{x^{38} \operatorname{Log}[x]}{1016878549747814369306915526592759332864000000} + \frac{x^{42} \operatorname{Log}[x]}{2870038018808231275931838382255403941075353600000000}$$

Approximate solution

Approximation by power series: fourth independed solution X_4

In[53]:= **zSol**

Out[53]=

$$A1\left(1-\frac{x^4}{64}+\frac{x^8}{147456}-\frac{x^{12}}{2123366400}+\frac{x^{16}}{106542032486400}-\frac{x^{20}}{13807847410237440000}+\frac{x^{24}}{3849406932415634472960000}-\frac{x^{28}}{2040124083669367620517232640000}+\frac{x^{32}}{1880178355509689199068681601024000000}-\frac{x^{36}}{2816838087944084125503921126295732224000000}+\frac{x^{40}}{6508022718386011963564259370193659730329600000000}\right)+$$

$$A2\left(x^2-\frac{x^6}{576}+\frac{x^{10}}{3686400}-\frac{x^{14}}{104044953600}+\frac{x^{18}}{8629904631398400}-\frac{x^{22}}{1670749536638730240000}+\frac{x^{26}}{650549771578242225930240000}-\frac{x^{30}}{459027918825607714616377344000000}+\frac{x^{34}}{543371544742300178530848982695936000000}-\frac{x^{38}}{1016878549747814369306915526592759332864000000}+\frac{x^{42}}{2870038018808231275931838382255403941075353600000000}\right)+$$

$$A3\left(\frac{3x^4}{128}-\frac{25x^8}{1769472}+\frac{49x^{12}}{42467328000}-\frac{761x^{16}}{29831769096192000}+\frac{7381x^{20}}{3479577547379834880000}-\frac{86021x^{24}}{1067055601665613875904512000000}+\frac{1171733x^{28}}{735179114791093315729589954150400000}-\frac{2436559x^{32}}{135508214438294319955278020349001728000000}+\frac{14274301x^{36}}{115041920982106750952880541934818740613939200000}-\frac{11167027x^{40}}{2020025692201651724251667551055159659189782970368000000}+\mathrm{Log}[x]-\frac{1}{64}x^4\,\mathrm{Log}[x]+\frac{x^8\,\mathrm{Log}[x]}{147456}-\frac{x^{12}\,\mathrm{Log}[x]}{2123366400}+\frac{x^{16}\,\mathrm{Log}[x]}{106542032486400}-\frac{x^{20}\,\mathrm{Log}[x]}{13807847410237440000}+\frac{x^{24}\,\mathrm{Log}[x]}{3849406932415634472960000}-\frac{x^{28}\,\mathrm{Log}[x]}{2040124083669367620517232640000}+\frac{x^{32}\,\mathrm{Log}[x]}{1880178355509689199068681601024000000}-\frac{x^{36}\,\mathrm{Log}[x]}{2816838087944084125503921126295732224000000}+\frac{x^{40}\,\mathrm{Log}[x]}{6508022718386011963564259370193659730329600000000}\right)+$$

$$A4\left(\frac{5x^6}{3456}-\frac{77x^{10}}{221184000}+\frac{223x^{14}}{14566293504000}-\frac{4609x^{18}}{2174735967112396800 0}+\frac{55991x^{22}}{4631317715562560225280000 0}-\frac{785633x^{26}}{2344321156859353685362212864 00000}+\frac{835397x^{30}}{1654153008279959960391577396838400 00000}-\frac{29889983x^{34}}{665751857535339993940280913974645489664 0000000}+\frac{197698279x^{38}}{789072536016270204785807637130921741871008972800 00000}-\frac{2736977x^{42}}{296943776753642803464995130005108469900898096644096 0000000}+x^2\,\mathrm{Log}[x]-\frac{1}{576}x^6\,\mathrm{Log}[x]+\frac{x^{10}\,\mathrm{Log}[x]}{3686400}-\frac{x^{14}\,\mathrm{Log}[x]}{104044953600}+\frac{x^{18}\,\mathrm{Log}[x]}{8629904631398400}-\frac{x^{22}\,\mathrm{Log}[x]}{1670749536638730240000}+\frac{x^{26}\,\mathrm{Log}[x]}{650549771578242225930240000}-\frac{x^{30}\,\mathrm{Log}[x]}{459027918825607714616377344000000}+\frac{x^{34}\,\mathrm{Log}[x]}{543371544742300178530848982695936000000}-\frac{x^{38}\,\mathrm{Log}[x]}{1016878549747814369306915526592759332864000000}+\frac{x^{42}\,\mathrm{Log}[x]}{2870038018808231275931838382255403941075353600000000}\right)$$

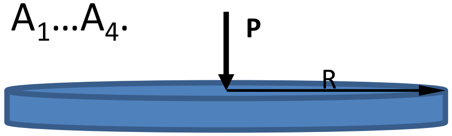
Approximate solution

Boundary conditions

The general solution, whose expression was defined by 4th approximate solutions, is reported below:

$$z = A_1 \cdot X_1(x) + A_2 \cdot X_2(x) + A_3 \cdot X_3(x) + A_4 \cdot X_4(x)$$

Four boundary conditions should be written for determining the four constants $A_1 \ldots A_4$.



The first two ones are imposed for r=R (namely, for x=R/l):

Dimensional expression

$$r = R$$

$$\left[\frac{d^2 w}{dr^2} + \frac{\nu}{r}\frac{dw}{dr}\right]_{r=R} = 0$$

$$\left[\frac{d}{dr}\left(\frac{d^2 w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)\right]_{r=R} = 0$$

Dimensionless expression

$$x = R/l$$

$$\left[\frac{d^2 z}{dx^2} + \frac{\nu}{x}\frac{dz}{dx}\right]_{x=\frac{R}{l}} = 0$$

$$\left[\frac{d}{dx}\left(\frac{d^2 z}{dx^2} + \frac{1}{x}\frac{dz}{dx}\right)\right]_{x=\frac{R}{l}} = 0$$

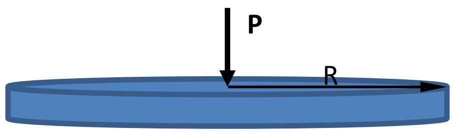
Approximate solution

Boundary conditions

The general solution, whose expression was defined by 4th approximate solutions, is reported below:

$$z = A_1 \cdot X_1(x) + A_2 \cdot X_2(x) + A_3 \cdot X_3(x) + A_4 \cdot X_4(x)$$

Other two boundary conditions are imposed for r=0 (namely, for x=0):



Dimensional expression

$$r = 0$$

$$\left[\frac{dw}{dr}\right]_{r=R} = 0$$

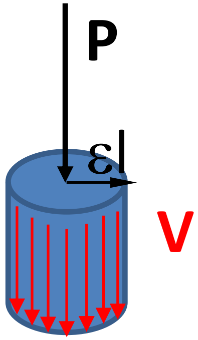
$$\lim_{\varepsilon -> 0}\left\{-2\pi\varepsilon l \cdot D\left[\frac{d}{dr}\left(\frac{d^2w}{dr^2} + \frac{1}{r}\frac{dw}{dr}\right)\right]_{r=\varepsilon l}\right\} + P = 0$$

Dimensionless expression

$$x = R/l$$

$$\left[\frac{dz}{dx}\right]_{x=0} = 0$$

$$\lim_{\varepsilon -> 0}\left\{-2\pi\varepsilon \cdot k_0 l^3\left[\frac{d}{dx}\left(\frac{d^2z}{dx^2} + \frac{1}{x}\frac{dz}{dx}\right)\right]_{x=\varepsilon}\right\} + P = 0$$



Approximate solution

Boundary conditions: expressions of constants

First of all, let's consider the third boundary condition:

$$\left[\frac{dz}{dx}\right]_{x=0}=0$$

Since X_1 and X_2 are polynomials with even order power terms (i.e. x^0, x^4, x^8 … and x^2, x^6, x^{10} …, respectively), their first derivative is zero for x=0.

Moreover, the first derivative of X_4 is also zero for x=0 (or, better, in the limit of $x \to 0$):

$$\left[\frac{dX_4}{dx}\right]_{x=0}=\lim_{\varepsilon\to 0}\left[\frac{d}{dx}\left(X_2\log x+F_4\right)\right]_{x=\varepsilon}=\lim_{\varepsilon\to 0}\left[2x\cdot\log x+\ldots-\frac{x^2}{x}+\ldots\right]_{x=\varepsilon}=0$$

Therefore, the first derivative of z in x=0 only takes a nonzero contribution from X_3:

$$\left[\frac{dz}{dx}\right]_{x=0}=A_3\cdot\left[\frac{dX_3}{dx}\right]_{x=0}=A_3\cdot\lim_{\varepsilon\to 0}\left[\frac{d}{dx}\left(X_1\log x+F_3\right)\right]_{x=\varepsilon}=A_3\cdot\lim_{\varepsilon\to 0}\left[-\frac{1}{x}+\ldots\right]_{x=\varepsilon}=0$$

which is only satisfied for $\underline{A_3=0}$.

Approximate solution

Boundary conditions: expressions of constants

Then, the fourth condition can be considered:

$$\lim_{\varepsilon \to 0} \left\{ -2\pi\varepsilon \cdot k_0 l^3 \left[\frac{d}{dx}\left(\frac{d^2 z}{dx^2} + \frac{1}{x}\frac{dz}{dx} \right) \right]_{x=\varepsilon} \right\} + P = 0$$

and, once again, no contribution comes from X_1 and X_2, but the only nonzero terms is given by the third derivatives of X_4:

$$\left[\frac{d}{dx}\left(\frac{d^2 z}{dx^2} + \frac{1}{x}\frac{dz}{dx} \right) \right]_{x=\varepsilon} = A_4 \cdot \left[\frac{d}{dx}\left(\frac{d^2 X_4}{dx^2} + \frac{1}{x}\frac{dX_4}{dx} \right) \right]_{x=\varepsilon} = A_4 \cdot \left[\frac{d}{dx}\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} \right)\left(X_2 \cdot \log x + F_4 \right) \right]_{x=\varepsilon} =$$

$$= A_4 \cdot \left[\frac{d}{dx}\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} \right)\left(X_2 \cdot \log x + F_4 \right) \right]_{x=\varepsilon} = A_4 \cdot \left[\frac{d}{dx}\left(\frac{d^2}{dx^2} + \frac{1}{x}\frac{d}{dx} \right)\left(x^2 \cdot \log x + \ldots \right) \right]_{x=\varepsilon} =$$

$$A_4 \cdot \left[\frac{2}{x} + \frac{2}{x} \right]_{x=\varepsilon} = A_4 \cdot \frac{4}{\varepsilon}$$

Therefore:

$$\lim_{\varepsilon \to 0} \left\{ -2\pi\varepsilon \cdot k_0 l^3 \cdot A_4 \cdot \frac{4}{\varepsilon} \right\} + P = 0 \quad \Rightarrow \quad -8\pi \cdot k_0 l^3 \cdot A_4 + P = 0 \quad \Rightarrow \quad \bar{A}_4 = \frac{P}{8\pi k_0 l^3}$$

Approximate solution

Boundary conditions: expressions of constants

Finally, the first two equations can be considered for determining the two constants A_1 and A_2:

$$\left[\frac{d^2z}{dx^2}+\frac{\nu}{x}\frac{dz}{dx}\right]_{x=\frac{R}{l}}=0 \qquad \left[\frac{d}{dx}\left(\frac{d^2z}{dx^2}+\frac{1}{x}\frac{dz}{dx}\right)\right]_{x=\frac{R}{l}}=0$$

$$\bar{z}(x)=\bar{A}_1\cdot X_1(x)+\bar{A}_2\cdot X_2(x)+\bar{A}_4\cdot X_4(x)$$

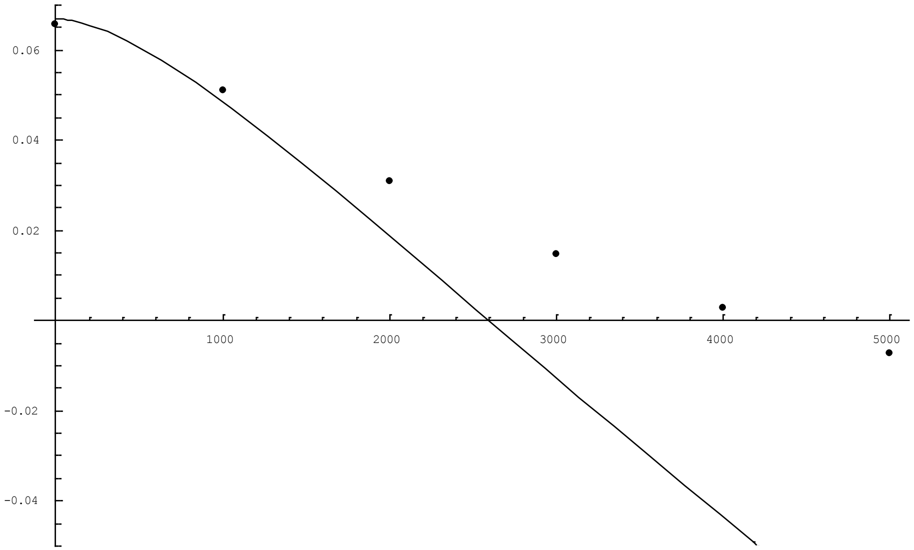
$$\bar{w}(r)=l\cdot\bar{z}\left(\frac{r}{l}\right)$$

$$\bar{M}_r(r)=-D\left(\frac{d^2\bar{w}}{dr^2}+\frac{\nu}{r}\frac{d\bar{w}}{dr}\right) \qquad \bar{M}_\theta(r)=-D\left(\frac{1}{r}\frac{d\bar{w}}{dr}+\nu\cdot\frac{d^2\bar{w}}{dr^2}\right) \qquad \bar{V}_r(r)=-D\left[\frac{d}{dr}\left(\frac{d^2\bar{w}}{dr^2}+\frac{1}{r}\frac{d\bar{w}}{dr}\right)\right]$$

Approximate solution

Comparisons: proposed procedure vs FEM solution

N_T=4



N_T=12