CIVIL AND ENVIRONMENTAL ENGINEERING

Elastic analysis of (concrete) plates on grade

Lesson 1: Mechanical background and numerical discretization

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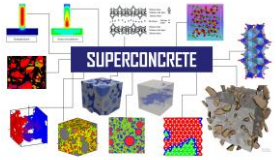

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Ann Arbor, MI - 20 October 2016

Overview

What is the problem we are going to face?

Determining the (generalised) stress fields in concrete slabs on grade (namely, plates on elastic soil) end subjected to sustained and live loads and thermal effects.

How will we approach it?

We will resume the formulation of the Kirchoff model for «thin» elastic plates, introduce the supporting effect of soil and convert the differential equations into their finite difference representation.

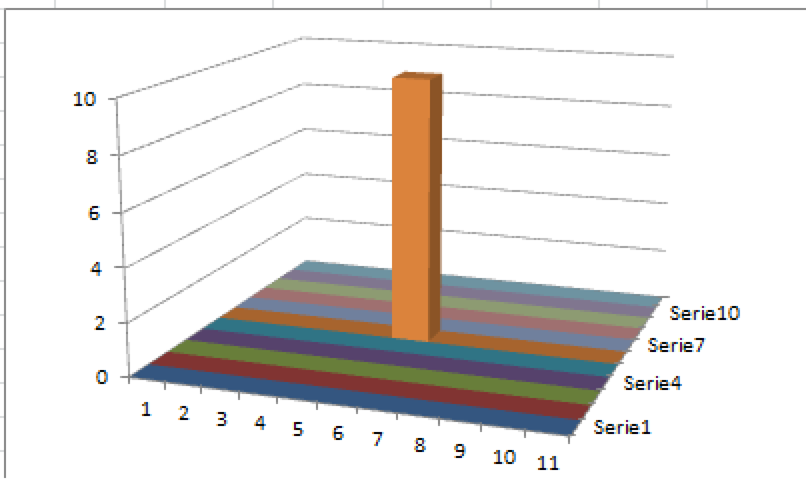
What is the final goal of this presentation?

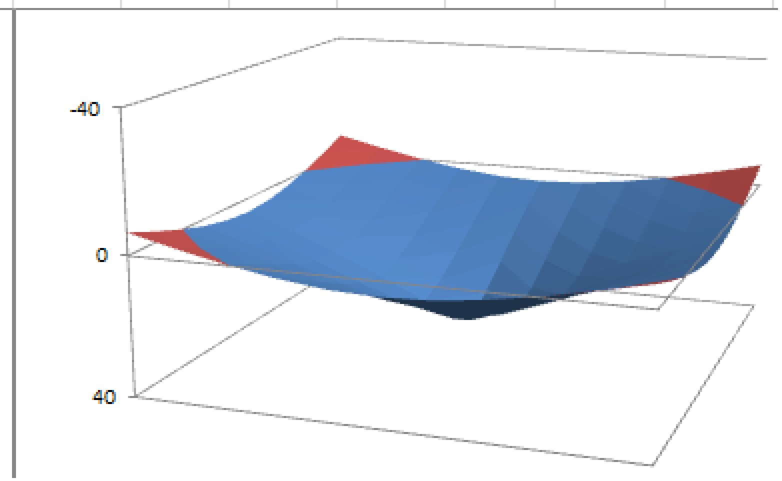
Realising a spreadsheet for analysing concrete slabs on elastic soil, subjected to self-weight, concentrated loads and thermal effects.

Overview

The final goal: a spreadsheet for the analysis of slabs on grade

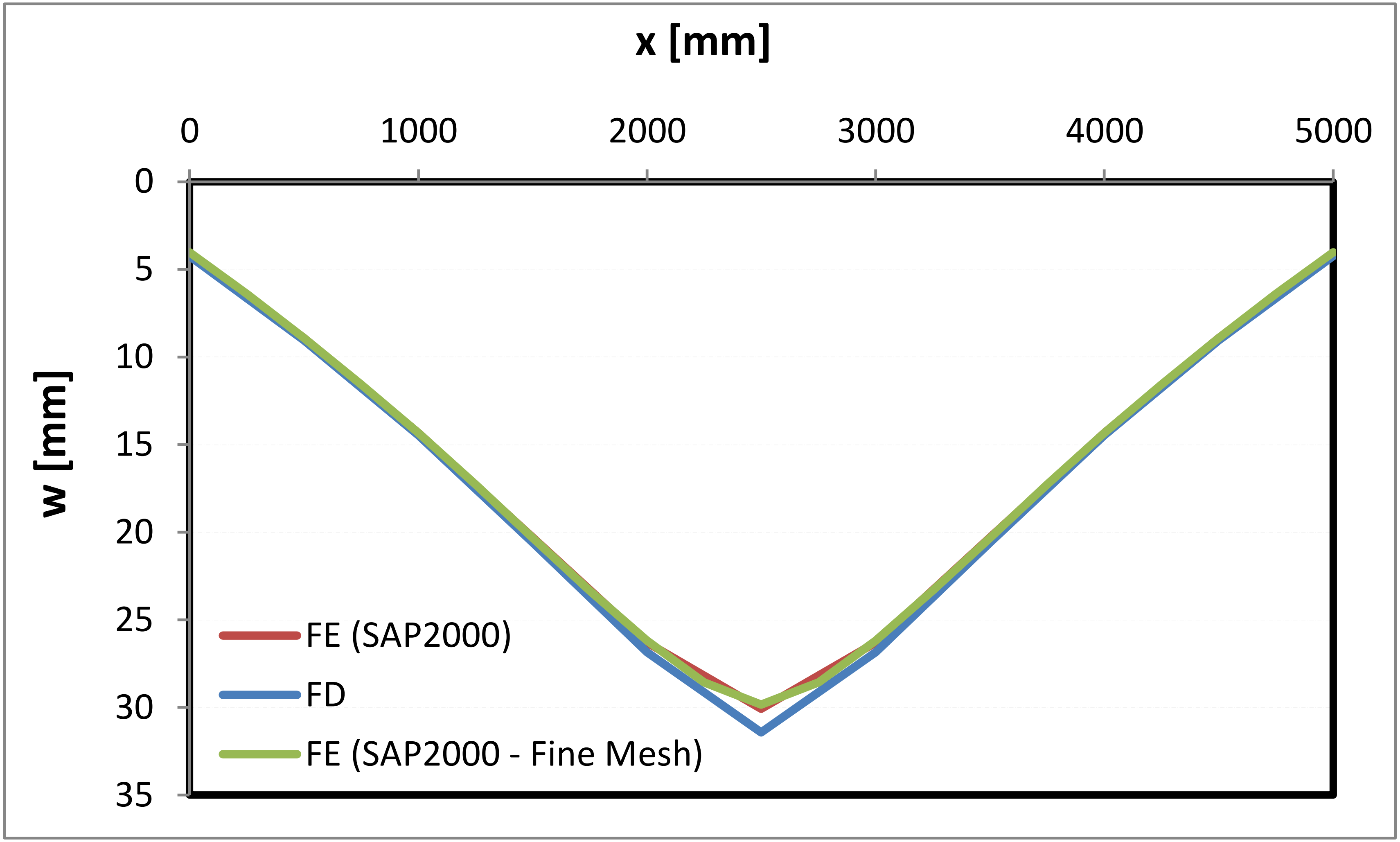
			-1.17E+01	-9.59E+00	-7.48E+00	-6.15E+00	-5.53E+00	-5.42E+00	-5.53E+00	-6.15E+00	-7.48E+00	-9.59E+00	-1.17E+01		
		-1.34E+01	-9.63E+00	-6.60E+00	-4.05E+00	-2.07E+00	-7.89E-01	-3.44E-01	-7.89E-01	-2.07E+00	-4.05E+00	-6.60E+00	-9.63E+00	-1.34E+01	
0	-1.17E+01	-9.63E+00	-6.43E+00	-3.24E+00	-3.56E-01	2.02E+00	3.65E+00	4.23E+00	3.65E+00	2.02E+00	-3.56E-01	-3.24E+00	-6.43E+00	-9.63E+00	-1.17E+01
500	-9.59E+00	-6.60E+00	-3.24E+00	1.83E-01	3.44E+00	6.27E+00	8.29E+00	9.04E+00	8.29E+00	6.27E+00	3.44E+00	1.83E-01	-3.24E+00	-6.60E+00	-9.59E+00
1000	-7.48E+00	-4.05E+00	-3.56E-01	3.44E+00	7.22E+00	1.07E+01	1.34E+01	1.45E+01	1.34E+01	1.07E+01	7.22E+00	3.44E+00	-3.56E-01	-4.05E+00	-7.48E+00
1500	-6.15E+00	-2.07E+00	2.02E+00	6.27E+00	1.07E+01	1.52E+01	1.89E+01	2.06E+01	1.89E+01	1.52E+01	1.07E+01	6.27E+00	2.02E+00	-2.07E+00	-6.15E+00
2000	-5.53E+00	-7.89E-01	3.65E+00	8.29E+00	1.34E+01	1.89E+01	2.40E+01	2.69E+01	2.40E+01	1.89E+01	1.34E+01	8.29E+00	3.65E+00	-7.89E-01	-5.53E+00
2500	-5.42E+00	-3.44E-01	4.23E+00	9.04E+00	1.45E+01	2.06E+01	2.69E+01	3.14E+01	2.69E+01	2.06E+01	1.45E+01	9.04E+00	4.23E+00	-3.44E-01	-5.42E+00
3000	-5.53E+00	-7.89E-01	3.65E+00	8.29E+00	1.34E+01	1.89E+01	2.40E+01	2.69E+01	2.40E+01	1.89E+01	1.34E+01	8.29E+00	3.65E+00	-7.89E-01	-5.53E+00
3500	-6.15E+00	-2.07E+00	2.02E+00	6.27E+00	1.07E+01	1.52E+01	1.89E+01	2.06E+01	1.89E+01	1.52E+01	1.07E+01	6.27E+00	2.02E+00	-2.07E+00	-6.15E+00
4000	-7.48E+00	-4.05E+00	-3.56E-01	3.44E+00	7.22E+00	1.07E+01	1.34E+01	1.45E+01	1.34E+01	1.07E+01	7.22E+00	3.44E+00	-3.56E-01	-4.05E+00	-7.48E+00
4500	-9.59E+00	-6.60E+00	-3.24E+00	1.83E-01	3.44E+00	6.27E+00	8.29E+00	9.04E+00	8.29E+00	6.27E+00	3.44E+00	1.83E-01	-3.24E+00	-6.60E+00	-9.59E+00
5000	-1.17E+01	-9.63E+00	-6.43E+00	-3.24E+00	-3.56E-01	2.02E+00	3.65E+00	4.23E+00	3.65E+00	2.02E+00	-3.56E-01	-3.24E+00	-6.43E+00	-9.63E+00	-1.17E+01
		-1.34E+01	-9.63E+00	-6.60E+00	-4.05E+00	-2.07E+00	-7.89E-01	-3.44E-01	-7.89E-01	-2.07E+00	-4.05E+00	-6.60E+00	-9.63E+00	-1.34E+01	
			-1.17E+01	-9.59E+00	-7.48E+00	-6.15E+00	-5.53E+00	-5.42E+00	-5.53E+00	-6.15E+00	-7.48E+00	-9.59E+00	-1.17E+01		





Overview

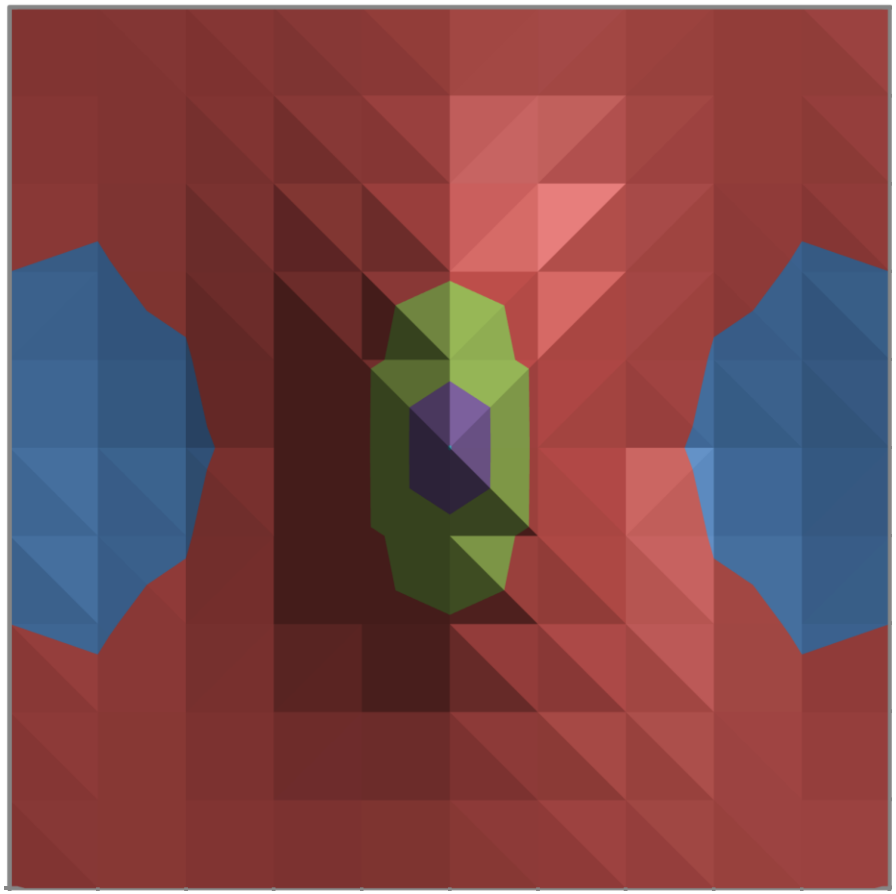
Comparisons with «black-box» commercial FE solutions/1



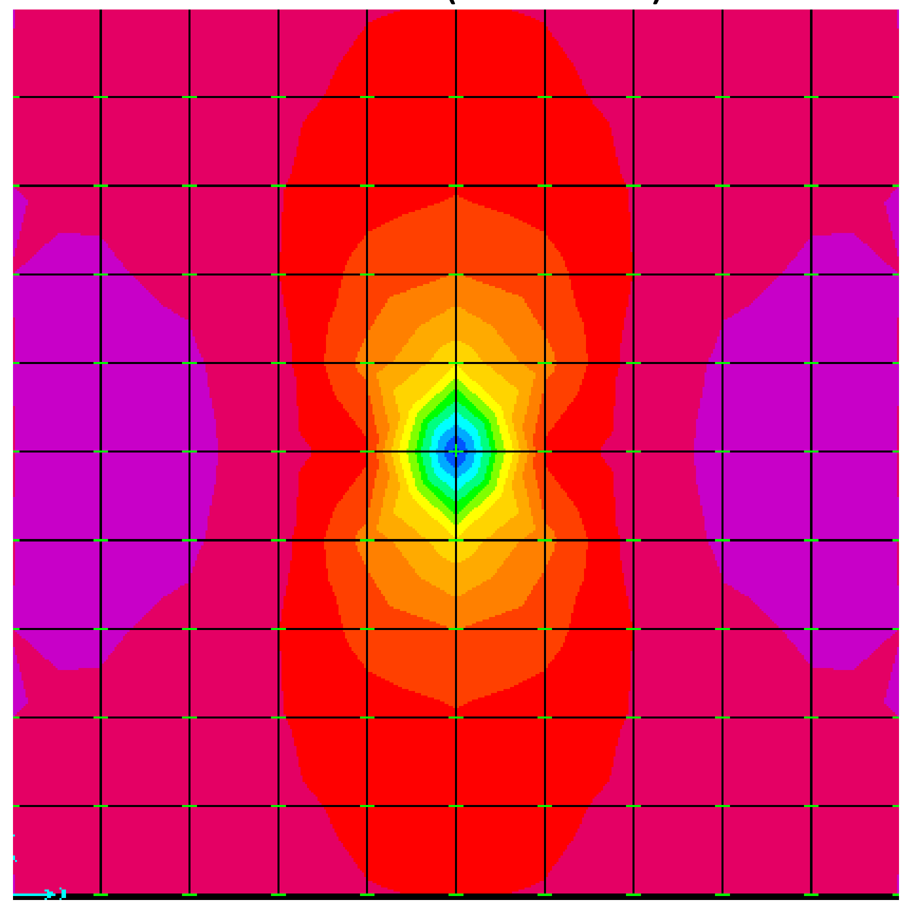
Overview

Comparisons with «black-box» commercial FE solutions/2

Finite Differences – M11



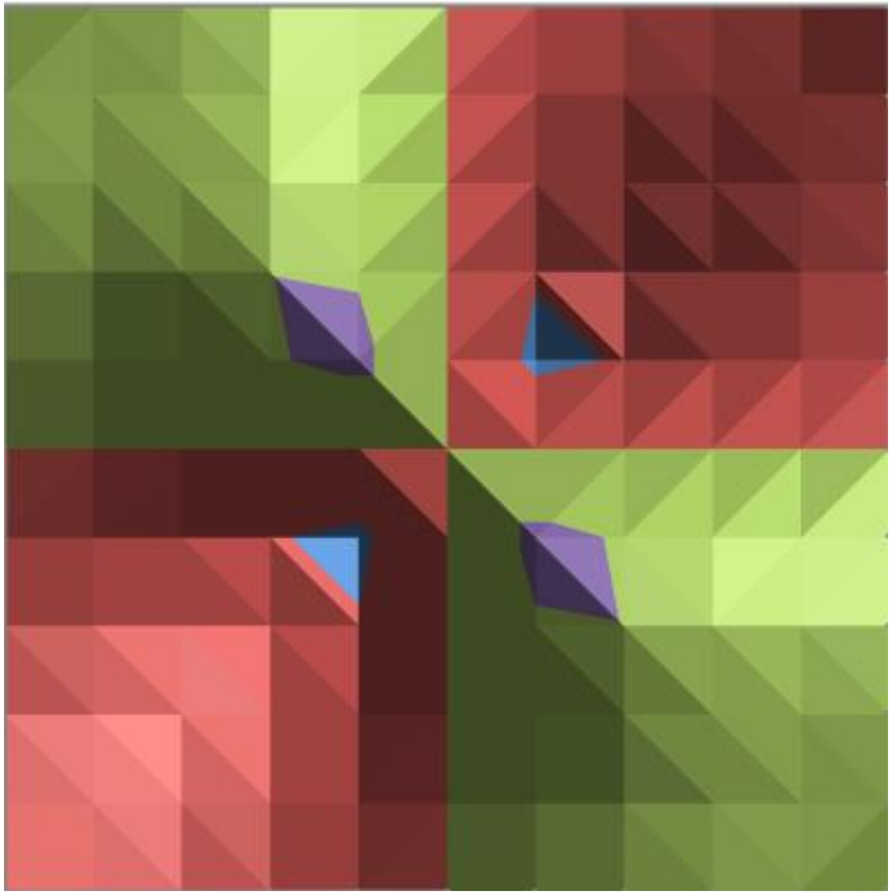
Finite Elements (SAP2000) – M11



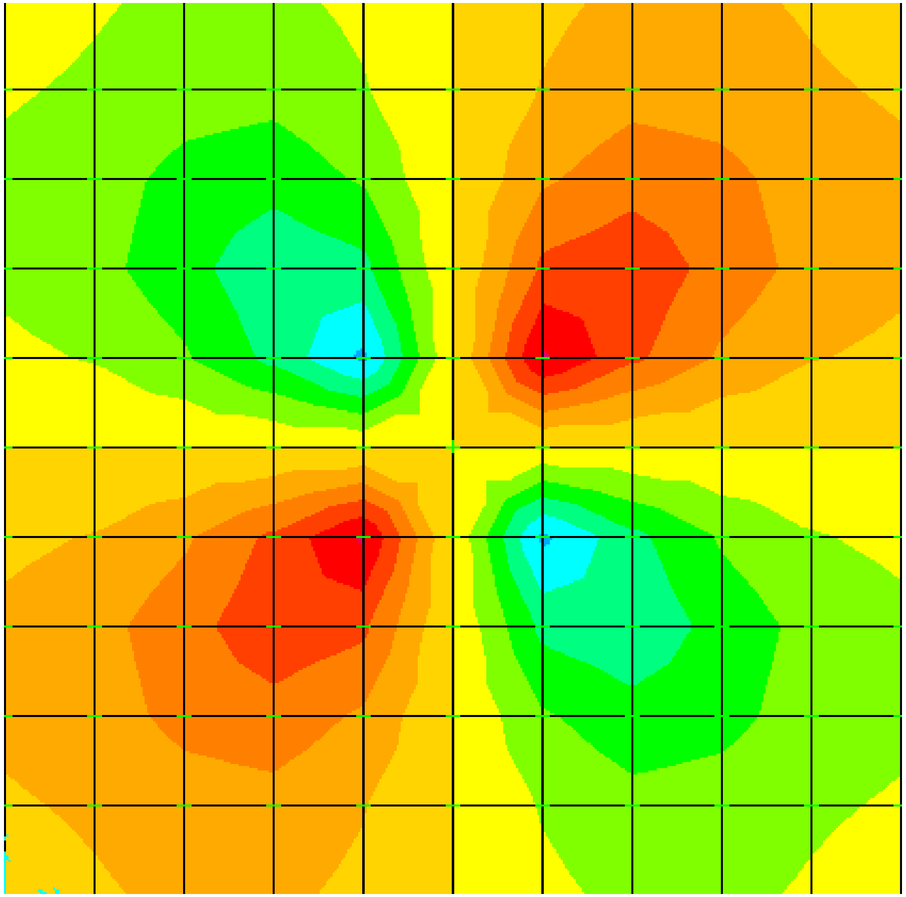
Overview

Comparisons with «black-box» commercial FE solutions/3

Differenze Finite – M12



Elementi Finiti (SAP2000) – M12



Overview

1. Thursday 20 October:
- Summary about the elastic theory of (thin) plates;
- Plates on grade (Winkler soil);
- Summary about Finite Difference (FD) schemes;
- Introduction to the FD solution of elastic plates on grade.

2. Tuesday 25 October:
- Clarifications about the previous lecture;
- Addition of thermal effects;
- Implementation in MS Excel;
- Examples and comparisons.

Overview

- ✓ Summary about the elastic theory of (thin) plates;
- ✓ Plates on grade (Winkler soil);
- ✓ Summary about Finite Difference (FD) schemes;
- ✓ FD solution of elastic plates on grade.

Theory of thin elastic plates

Kirchhoff–Love plate theory: definitions

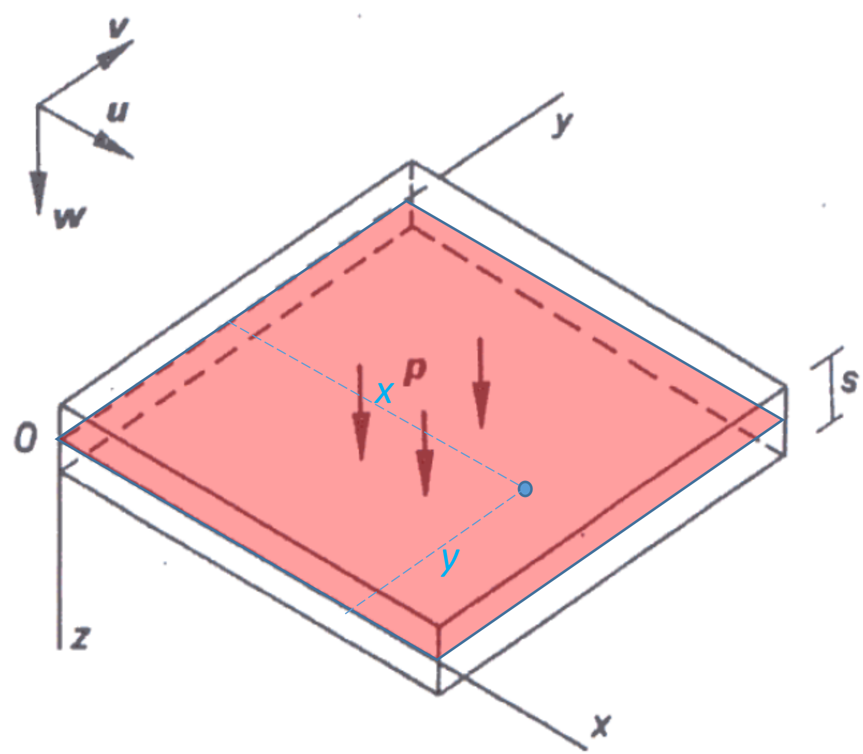
Plates are defined as structural elements with a "small" thickness compared to the planar dimensions.

Thickness-to-width ratio of plates is less than 0.1.

Plate theories are based on this "disparity" in lengths, which allows reducing the full 3D continuum mechanics problem into a 2D problem.

Plate theories aim is to calculate the displacement and stresses fields induced by external actions.

The Kirchhoff–Love theory for "plates" is the equivalent of Euler–Bernoulli for "beams".



*It is assumed that a **mid-surface plane** can be used to represent the 3D into 2D.*

$$w = w(x, y)$$

Theory of thin elastic plates

Kirchhoff–Love plate theory: assumptions

The following kinematic <u>assumptions</u> that are made in this theory:

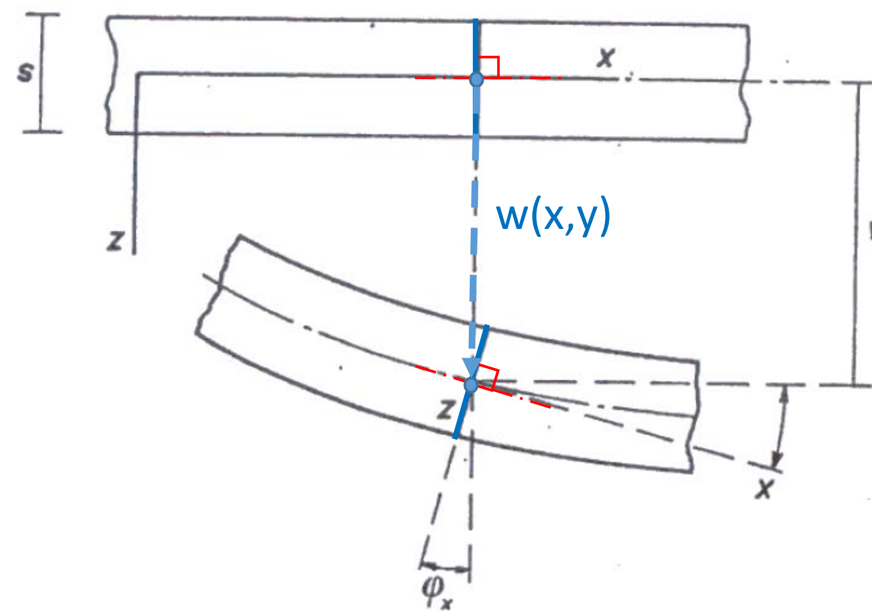
- ✓ *straight lines normal to the mid-surface ("chords") remain straight after deformation;*
- ✓ *straight lines normal to the mid-surface remain normal to the mid-surface after deformation;*
- ✓ *the thickness of the plate does not change as a results of deformation.*

Main Kirchhoff–Love assumption:

$$\varphi_x = -\frac{\partial w}{\partial x}$$

N.B.: φ_x is the rotation of the normal segment in the x-z plane

Undeformed configuration



Deformed configuration

Theory of thin elastic plates

Kirchhoff–Love plate theory: displacement field

As a consequence, the displacement of any point of the "chord" passing through the point (x,y) of the mid-surface depends on the rotation φ_x *and the distance z from the mid-surface:*

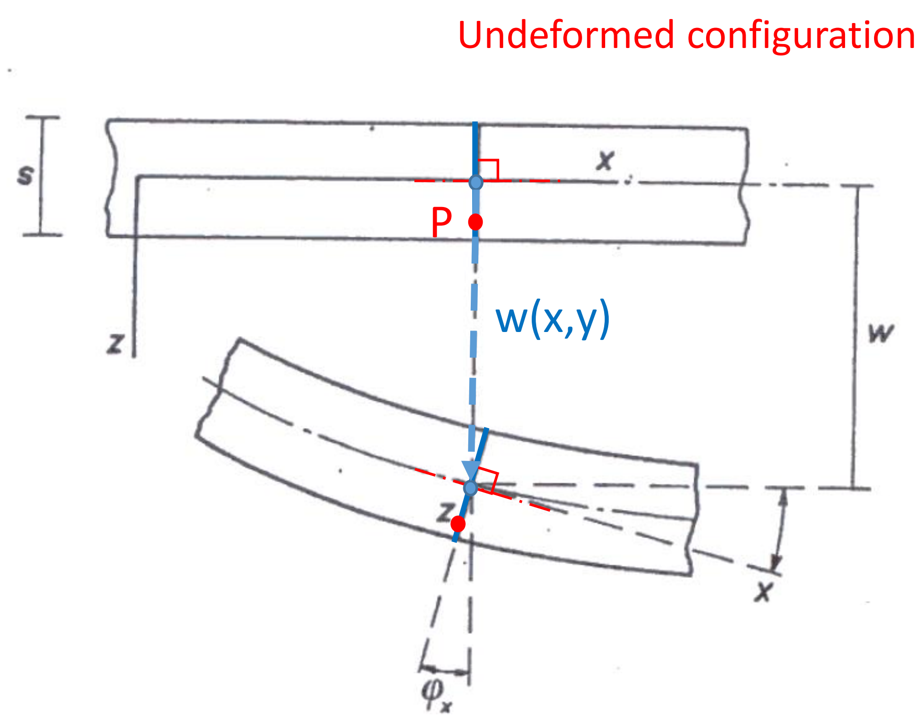
$$u = -z \cdot \frac{\partial w}{\partial x}$$

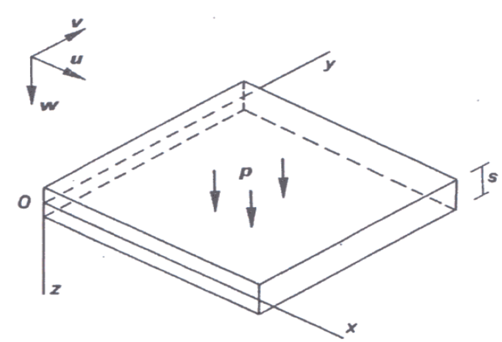
Similarly:

$$v = -z \cdot \frac{\partial w}{\partial y}$$

The displacements of a point P(x,y,z) depends on:

- ✓ *the displacement w(x,y) of the mid-surface;*
- ✓ *the distance z from the mid-surface.*





Theory of thin elastic plates

Kirchhoff–Love plate theory: strain field

The six components *<u>strain field</u>* *can be determined from the* *<u>displacement field</u>* *based on the general* *<u>compatibility conditions</u>* *commonly assumed in continuum mechanics:*

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z \cdot \frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z \cdot \frac{\partial^2 w}{\partial y^2}$$

Axial strain on the mid-surface (z=0) are zero

$$\varepsilon_z = \frac{\partial w}{\partial z} = 0$$

Assumption #3: the thickness of the plate does not change as a results of deformation

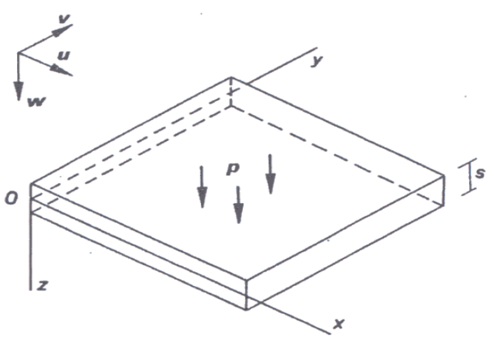
$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z \cdot \frac{\partial^2 w}{\partial x \partial y}$$

Shear strain on the mid-surface (z=0) are zero

$$\gamma_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} = 0$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = 0$$

The straight lines remain perpendicular to the deformed mid-surface



Theory of thin elastic plates

Kirchhoff–Love plate theory: stress field

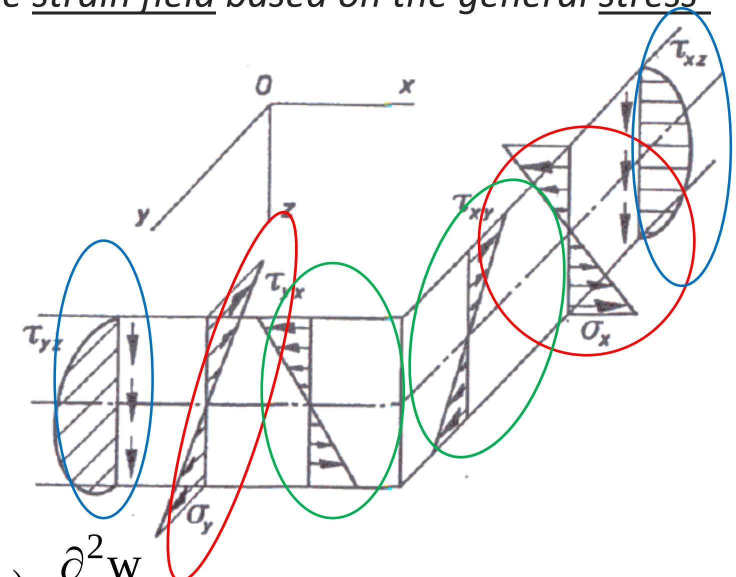
The six components <u>stress field</u> can be <u>partly</u> determined from the <u>strain field</u> based on the general <u>stress-strain relationships</u> assumed in continuum mechanics:

$$\sigma_x = \frac{E}{1-\nu^2}\cdot\left(\varepsilon_x + \nu\cdot\varepsilon_y\right) = -\frac{E\cdot z}{1-\nu^2}\cdot\left(\frac{\partial^2 w}{\partial x^2} + \nu\cdot\frac{\partial^2 w}{\partial y^2}\right)$$

$$\sigma_y = \frac{E}{1-\nu^2}\cdot\left(\varepsilon_y + \nu\cdot\varepsilon_x\right) = -\frac{E\cdot z}{1-\nu^2}\cdot\left(\frac{\partial^2 w}{\partial y^2} + \nu\cdot\frac{\partial^2 w}{\partial x^2}\right)$$

$$\sigma_z \approx 0$$

$$\tau_{xy} = G\cdot\gamma_{xy} = -2z\cdot\frac{E}{2\cdot(1-\nu)}\cdot\frac{\partial^2 w}{\partial x\partial y} = -z\cdot\frac{E}{\left(1-\nu^2\right)}\cdot(1+\nu)\cdot\frac{\partial^2 w}{\partial x\partial y}$$



The components τ_{xz} and τ_{yz} can only be determined from equilibrium conditions:

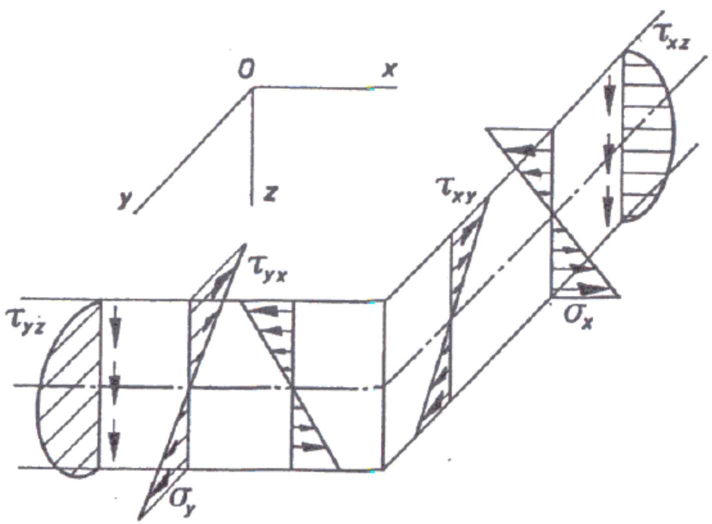
$$\frac{\partial\sigma_x}{\partial x} + \frac{\partial\tau_{xy}}{\partial y} + \frac{\partial\tau_{xz}}{\partial z} = 0 \qquad \tau_{xz} = \frac{E}{1-\nu^2}\cdot\left(\frac{s^2}{8} - \frac{z^2}{2}\right)\cdot\frac{\partial}{\partial x}\Delta w$$

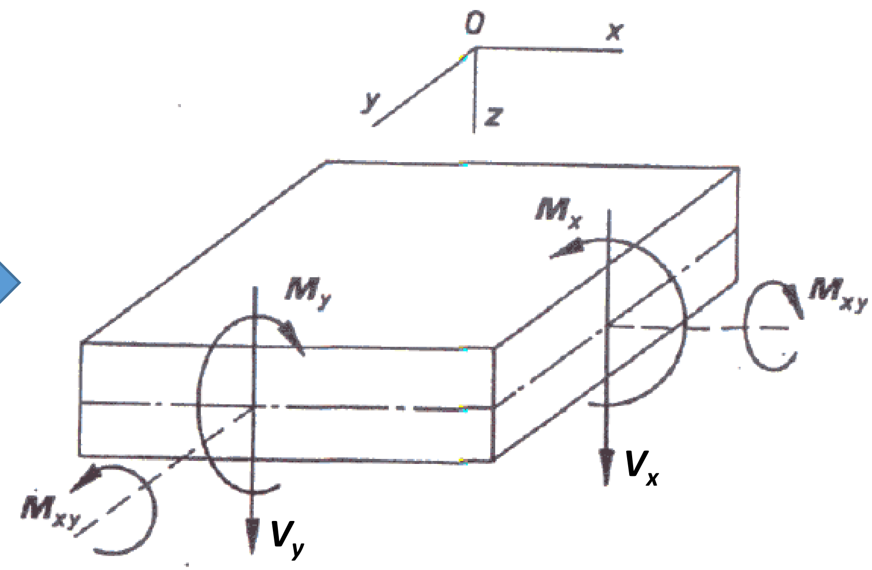
$$\frac{\partial\tau_{yx}}{\partial x} + \frac{\partial\sigma_y}{\partial y} + \frac{\partial\tau_{yz}}{\partial z} = 0 \qquad \tau_{yz} = \frac{E}{1-\nu^2}\cdot\left(\frac{s^2}{8} - \frac{z^2}{2}\right)\cdot\frac{\partial}{\partial y}\Delta w$$

$$\Delta w = \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}$$

Theory of thin elastic plates

Kirchhoff–Love plate theory: generalised stress field/1





$$\sigma_x = -\frac{E \cdot z}{1-\nu^2} \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} \right)$$

$$\sigma_y = -\frac{E \cdot z}{1-\nu^2} \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2} \right)$$

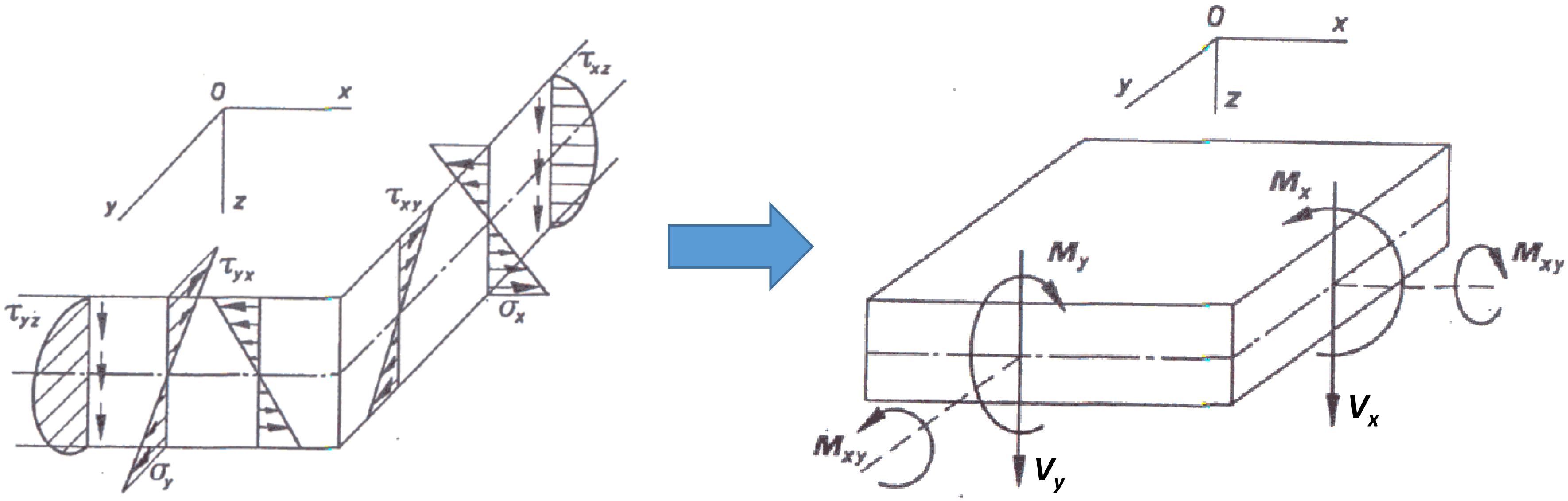
$$M_x = \int_{-s/2}^{s/2} \sigma_x \cdot z \cdot dz = -D \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} \right)$$

$$M_y = \int_{-s/2}^{s/2} \sigma_y \cdot z \cdot dz = -D \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2} \right)$$

$$D = \frac{E \cdot s^3}{12 \cdot \left(1-\nu^2\right)}$$

Theory of thin elastic plates

Kirchhoff–Love plate theory: generalised stress field/2



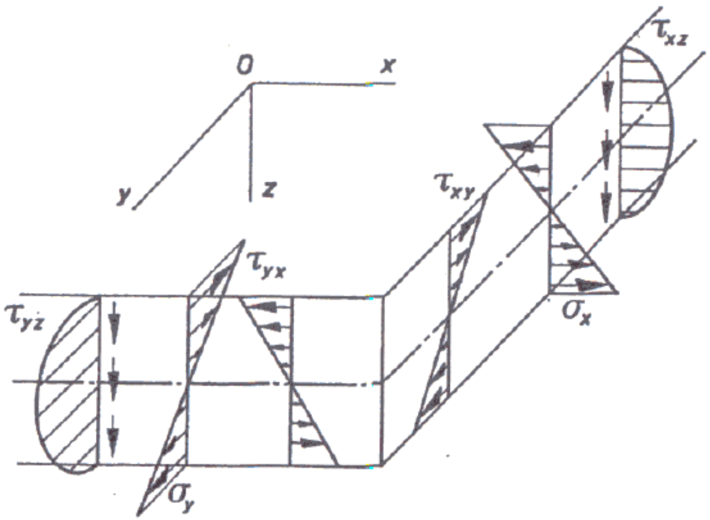
$$\tau_{xy} = -z \cdot \frac{E}{\left(1-\nu^2\right)} \cdot (1+\nu) \cdot \frac{\partial^2 w}{\partial x \partial y}$$

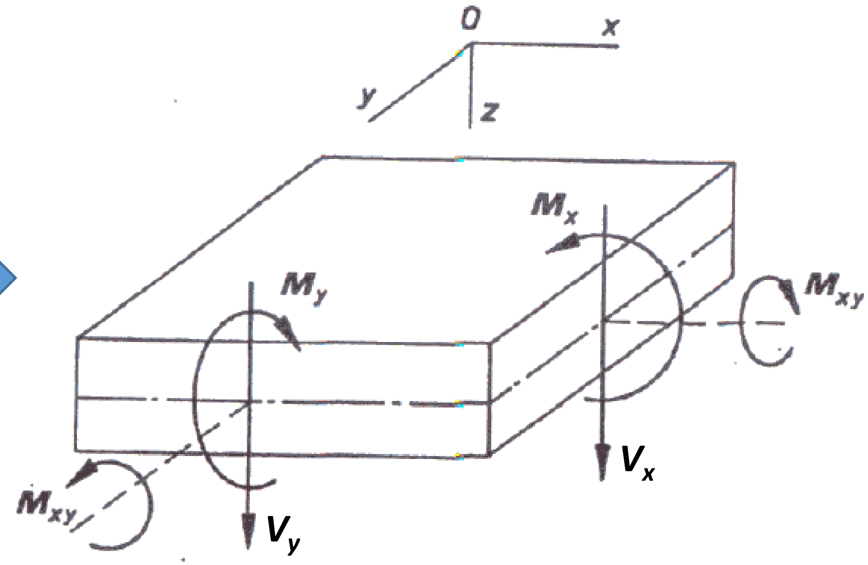
$$M_{xy} = \int_{-s/2}^{s/2} \tau_{xy} \cdot z \cdot dz = -D \cdot (1-\nu) \cdot \frac{\partial^2 w}{\partial x \partial y}$$

$$D = \frac{E \cdot s^3}{12 \cdot \left(1-\nu^2\right)}$$

Theory of thin elastic plates

Kirchhoff–Love plate theory: generalised stress field/3





$$\tau_{xz} = \frac{E}{1-\nu^2} \cdot \left(\frac{s^2}{8} - \frac{z^2}{2} \right) \cdot \frac{\partial}{\partial x} \Delta w$$

$$\tau_{yz} = \frac{E}{1-\nu^2} \cdot \left(\frac{s^2}{8} - \frac{z^2}{2} \right) \cdot \frac{\partial}{\partial y} \Delta w$$

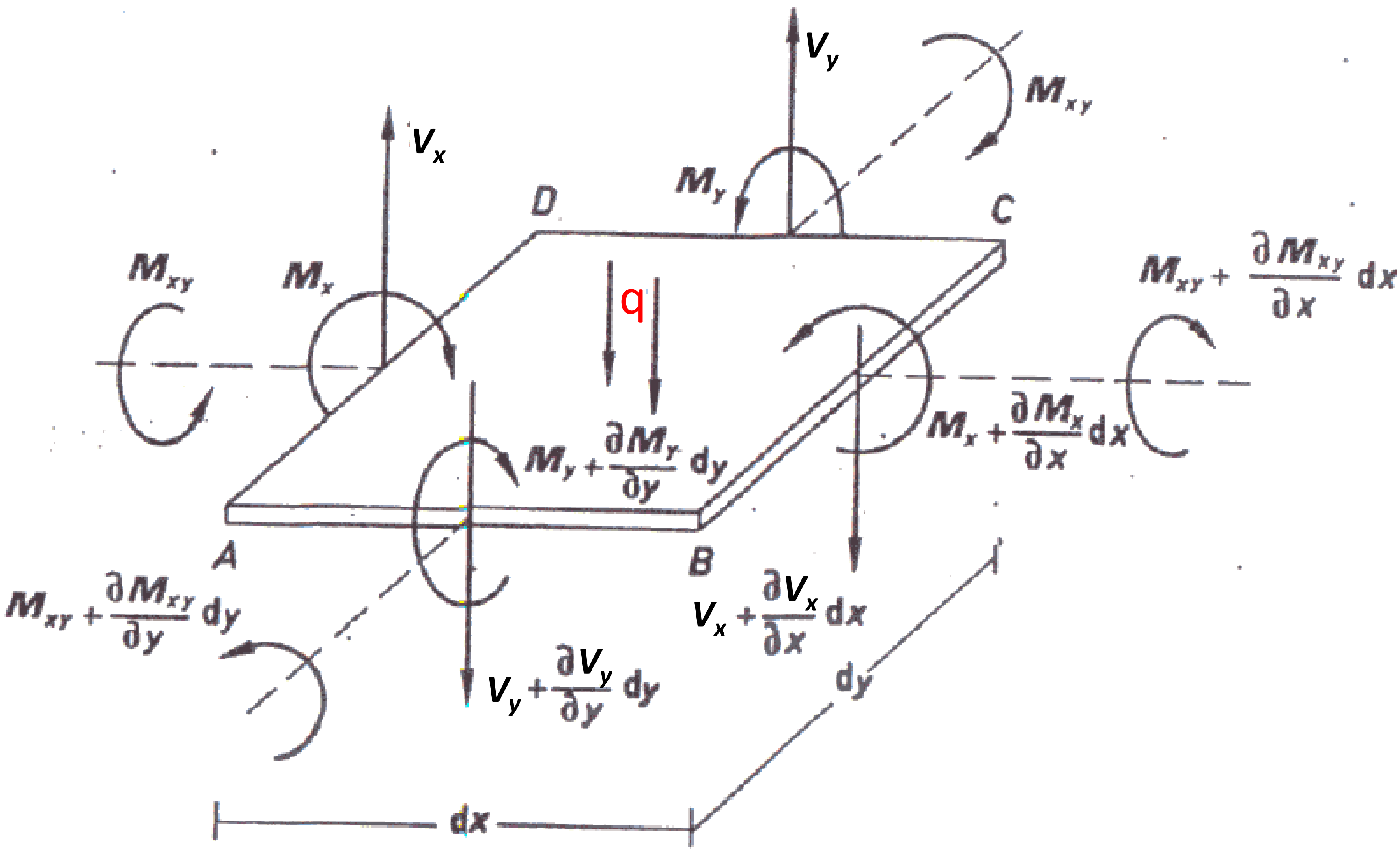
$$V_x = \int_{-s/2}^{s/2} \tau_{xz} \cdot dz = -D \cdot \frac{\partial}{\partial x} \Delta w$$

$$V_y = \int_{-s/2}^{s/2} \tau_{yz} \cdot dz = -D \cdot \frac{\partial}{\partial y} \Delta w$$

$$D = \frac{E \cdot s^3}{12 \cdot \left(1-\nu^2\right)}$$

Theory of thin elastic plates

Kirchhoff–Love plate theory: equilibrium equations/1



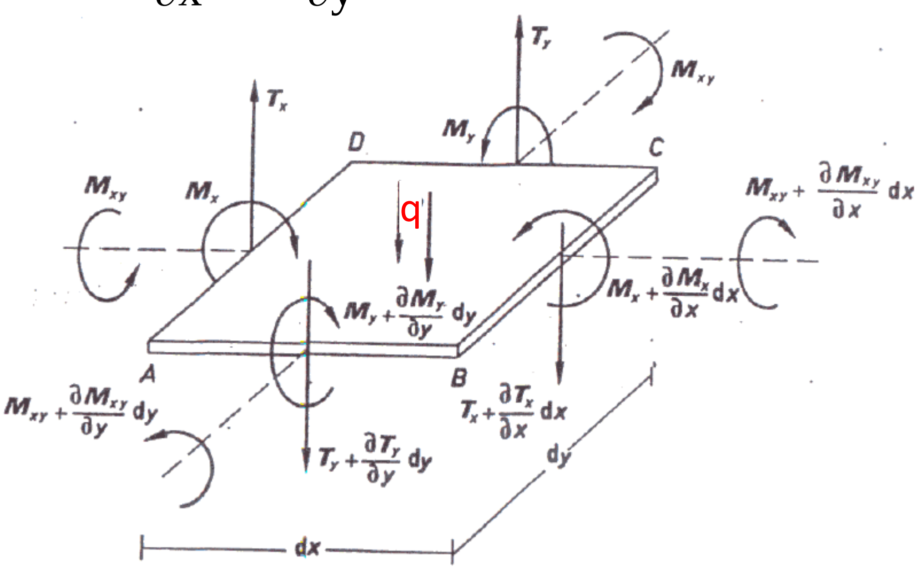
Theory of thin elastic plates

Kirchhoff–Love plate theory: equilibrium equations/3

$$\frac{\partial V_x}{\partial x}+\frac{\partial V_y}{\partial y}+q=0$$

$$\frac{\partial M_x}{\partial x}+\frac{\partial M_{xy}}{\partial y}-V_x=0$$

$$\frac{\partial M_{xy}}{\partial x}+\frac{\partial M_y}{\partial y}-V_y=0$$



$$\frac{\partial^2 M_x}{\partial x^2}+2\cdot\frac{\partial^2 M_{xy}}{\partial x\partial y}+\frac{\partial^2 M_y}{\partial y^2}+q=0$$

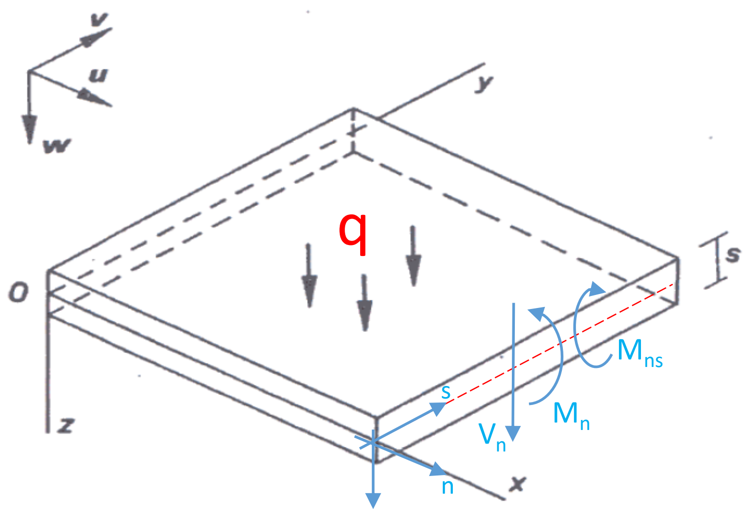
$$\frac{\partial^4 w}{\partial x^4}+2\cdot\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{q}{D}$$

Germain-Lagrange equation

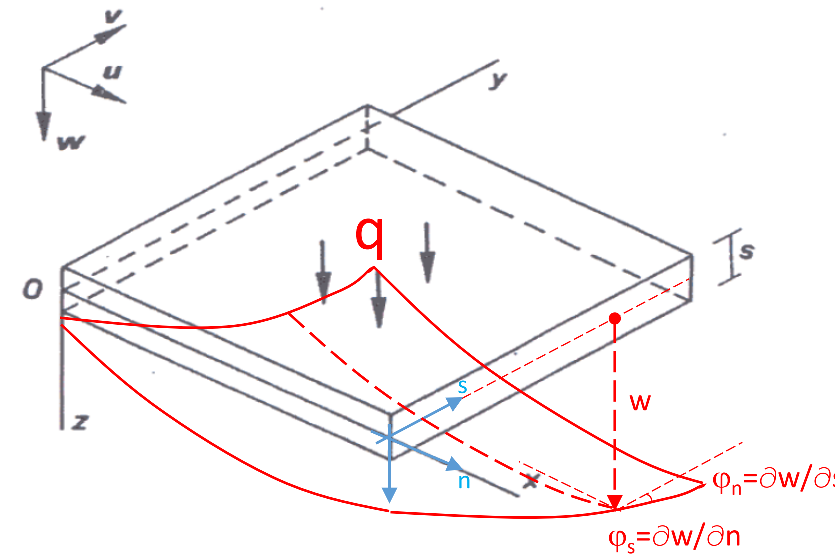
Theory of thin elastic plates

Kirchhoff–Love plate theory: Boundary Conditions/1

Force Components



Displacement Components



V_n normal shear force

M_n (normal) bending moment

M_{ns} (tangential) torsional moment

w transverse displacement

φ_s rotation in the plane s-z

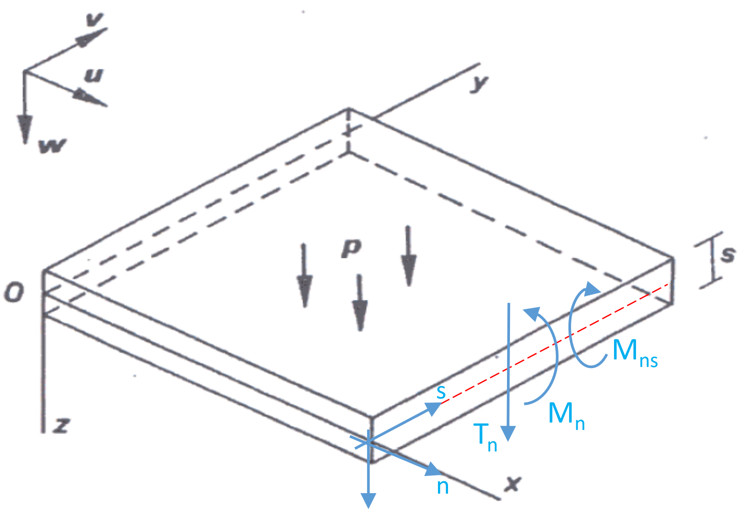
φ_n rotation in the plane n-z

Rotations are not independent of the transverse displacement w

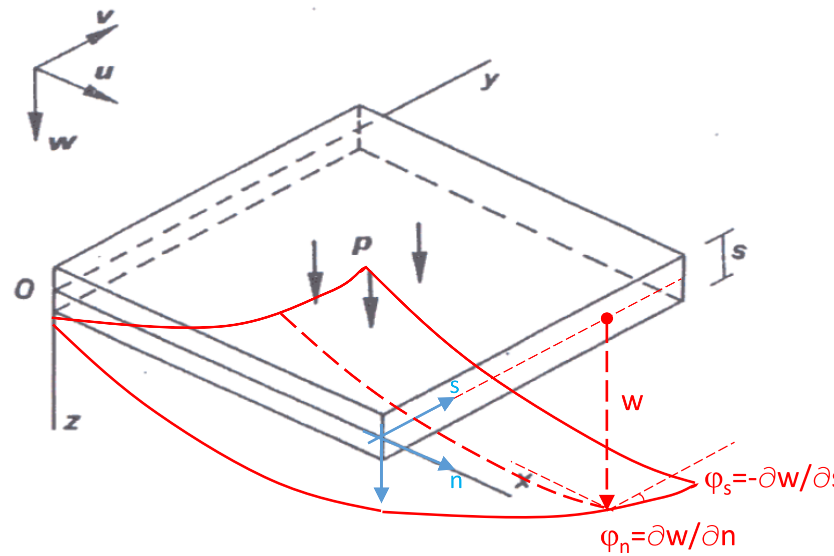
Theory of thin elastic plates

Kirchhoff–Love plate theory: Boundary Conditions/2

Force Components



Displacement Components



$$W = \int_{\partial P} M_n \cdot \varphi_n \cdot ds + \int_{\partial P} M_{ns} \cdot \varphi_s \cdot ds + \int_{\partial P} V_n \cdot w \cdot ds = 0$$

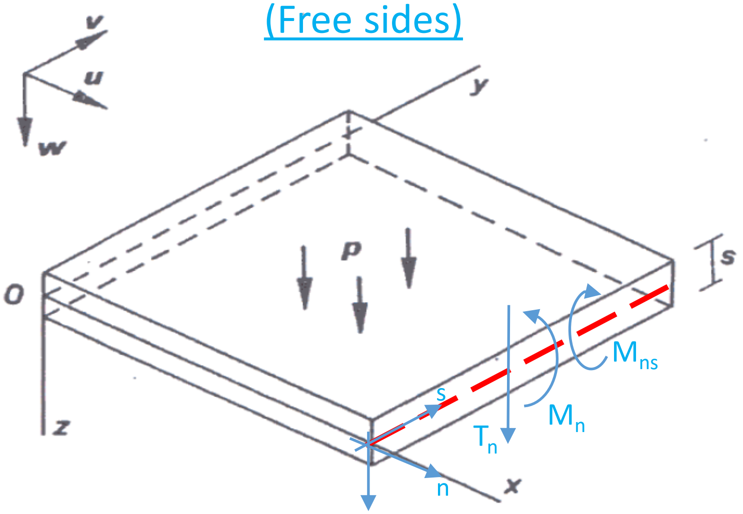
$$W = \int_{\partial P} M_n \cdot \frac{\partial w}{\partial n} \cdot ds + \int_{\partial P} M_{ns} \cdot \frac{\partial w}{\partial s} \cdot ds + \int_{\partial P} V_n \cdot w \cdot ds =$$

$$= \int_{\partial P} M_n \cdot \frac{\partial w}{\partial n} \cdot ds + \int_{\partial P} \underbrace{\boxed{\left(V_n + \frac{\partial M_{ns}}{\partial s}\right)}}_{R_n} \cdot w \cdot ds - \sum_{i=1}^{n_{edges}} \left(M_{ns,i}'' - M_{ns,i}'\right) \cdot w_i = 0$$

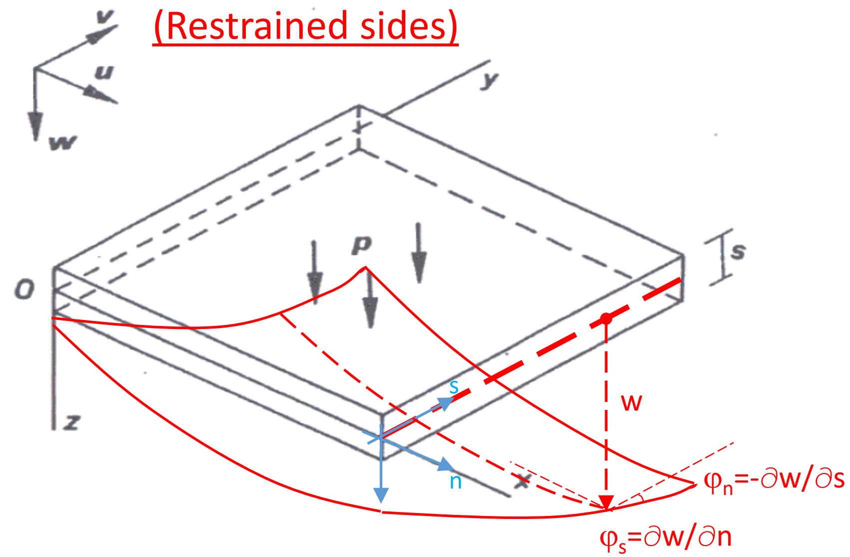
Theory of thin elastic plates

Kirchhoff–Love plate theory: Boundary Conditions/3

Static Conditions
(Free sides)



Kinematic Conditions
(Restrained sides)



$$R_x = V_x + \frac{\partial M_{xy}}{\partial y} = 0$$

$$2M_{xy}\Big|_{\substack{x=L_x \\ y=L_y}} = 0$$

or

$$w = 0$$

$$M_x = 0$$

or

$$\frac{\partial w}{\partial x} = 0$$

Theory of thin elastic plates

Kirchhoff–Love plate theory: summary

Beams in bending (Euler-Bernoulli Theory)

Assumptions*: transverse plane sections remain plane and normal to the longitudinal axis after deformation.*

Differential equation*:*

$$\frac{d^4 w}{dx^4} = \frac{q}{EI}$$

Boundary conditions*:*

$$w = 0 \quad \textit{or} \quad V = 0$$

$$\frac{dw}{dx} = 0 \quad \textit{or} \quad M = 0$$

Plates in bending (Kirchhoff-Love Theory)

Assumptions*: normal straight segments remain straight and normal to the mid-surface after deformation.*

Differential equation*:*

$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Boundary conditions*:*

$$w = 0 \quad \textit{or} \quad R = 0, \; M_{xy,edges} = 0$$

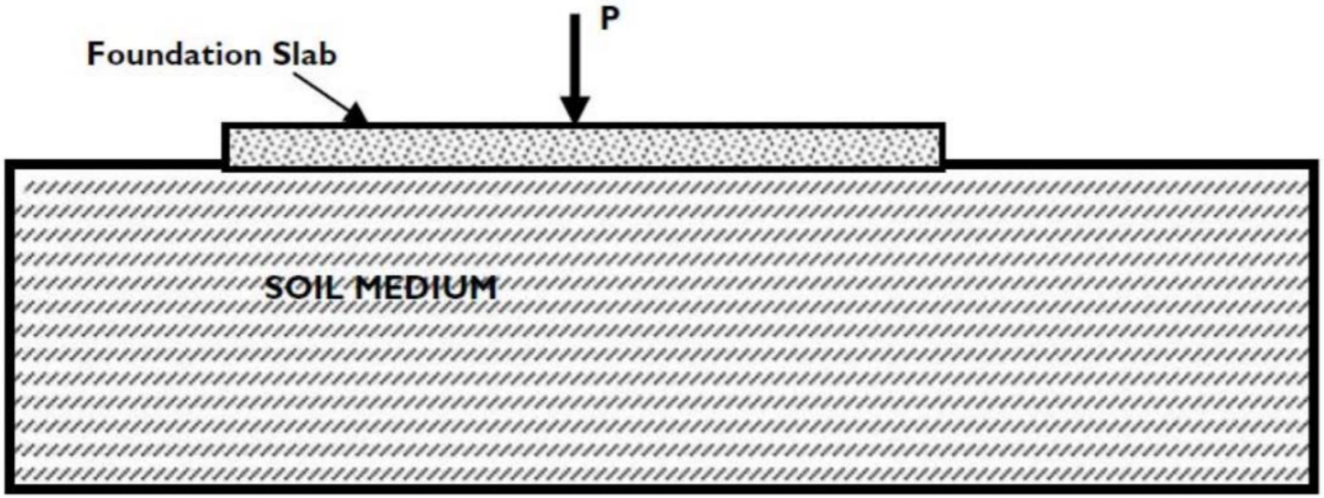
$$\frac{\partial w}{\partial n} = 0 \quad \textit{or} \quad M_n = 0$$

Overview

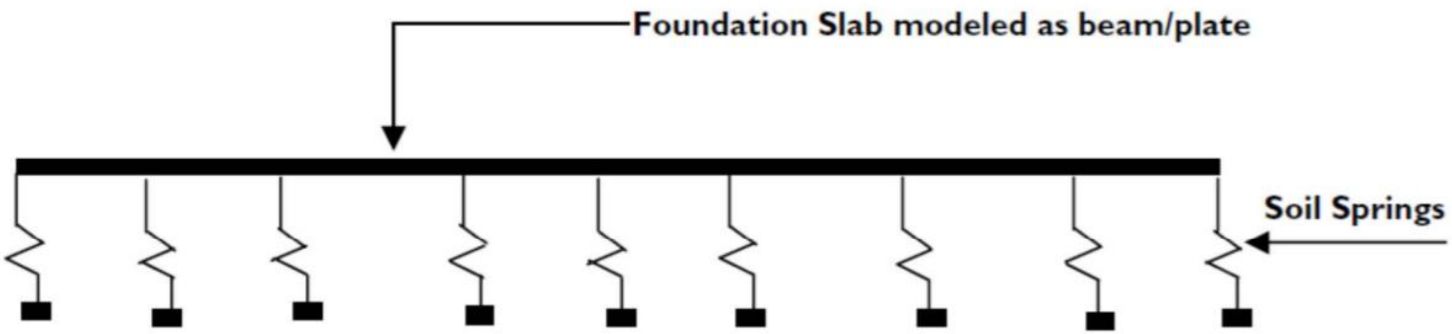
- ✓ Summary about the elastic theory of (thin) plates;
- ✓ Plates on grade (Winkler soil);
- ✓ Summary about Finite Difference (FD) schemes;
- ✓ FD solution of elastic plates on grade.

Plates on grade

Winkler soil model



$$\sigma_s = k0w$$



According to the Winkler's model the soil behaves as a bad of mutually independent linear (and bi-lateral) springs.

Plates on grade

Winkler soil model: advantages and limitations

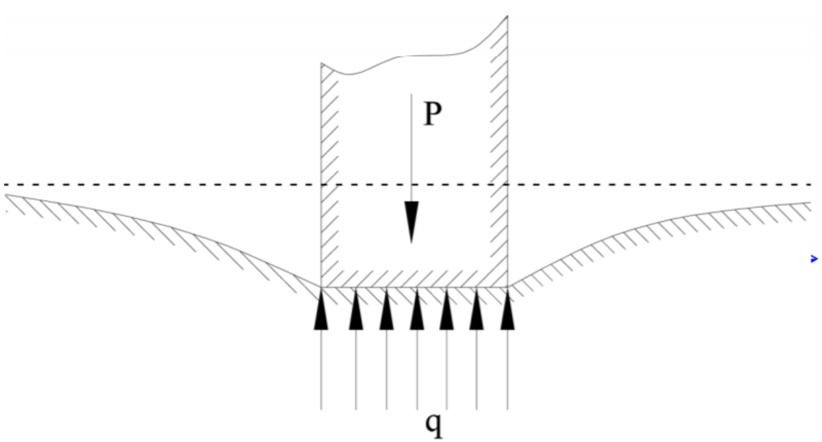
Advantage: the soil reaction $\sigma_s(x,y)$ in a given point of coordinates (x,y) only depends on the local vertical displacement w(x,y) multiplied by a stiffness coefficient (FL^{-3}] often referred to as "modulus of subgrade".

$$\sigma_f(x,y) = k_0 \cdot w(x,y)$$

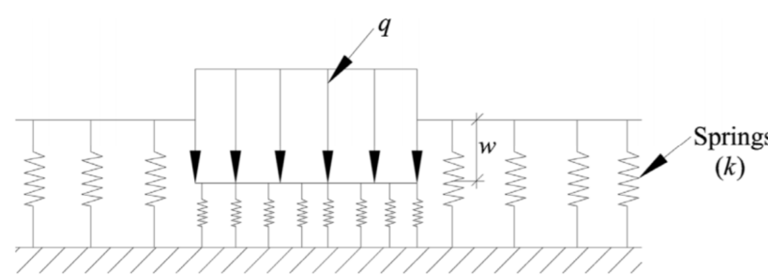
$$w(x,y) = \frac{\sigma_f(x,y)}{k_0}$$

Limitations:

Actual response

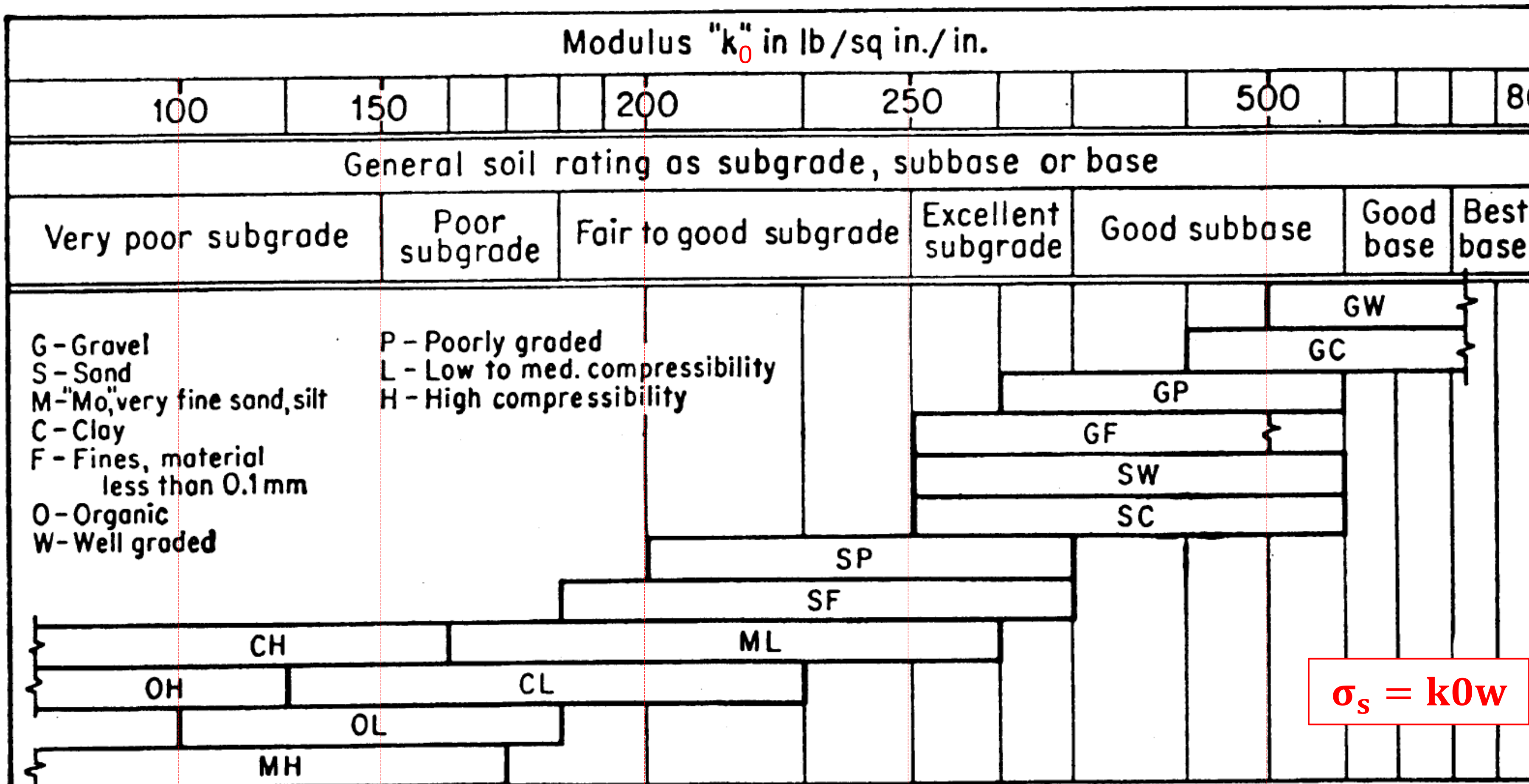


Winkler idealisation



Plates on grade

Winkler soil model: modulus of subgrade k_0

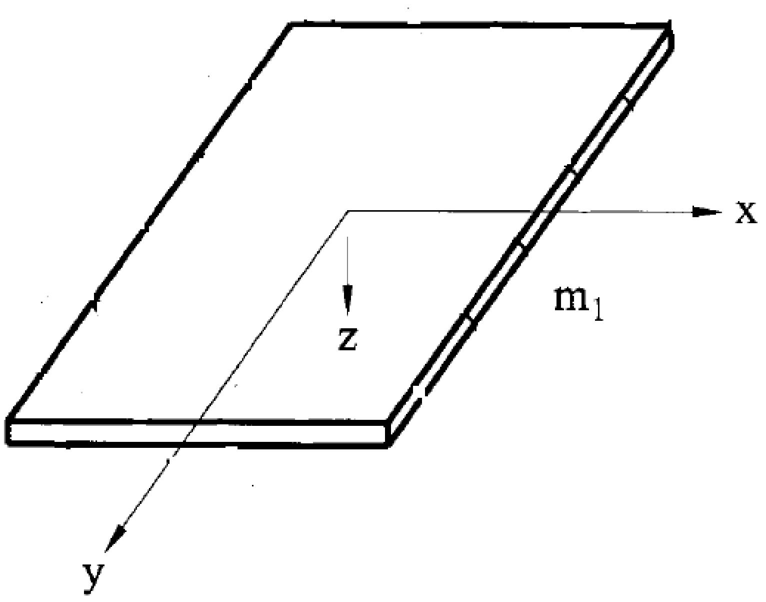


N/mm^3	0.0271	0.0407	0.0543	0.0679	0.136
kg_f/cm^3	2.76	4.15	5.53	6.92	13.83

Fundamental equations

Kirchhoff theory of thin plates

$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$



$q = q(x, y)$ Normal load function

$w = w(x, y)$ Normal displacement function

Equation of plates on an elastic foundation

Winkler soil

$$\sigma = -k_0 \cdot w(x, y)$$

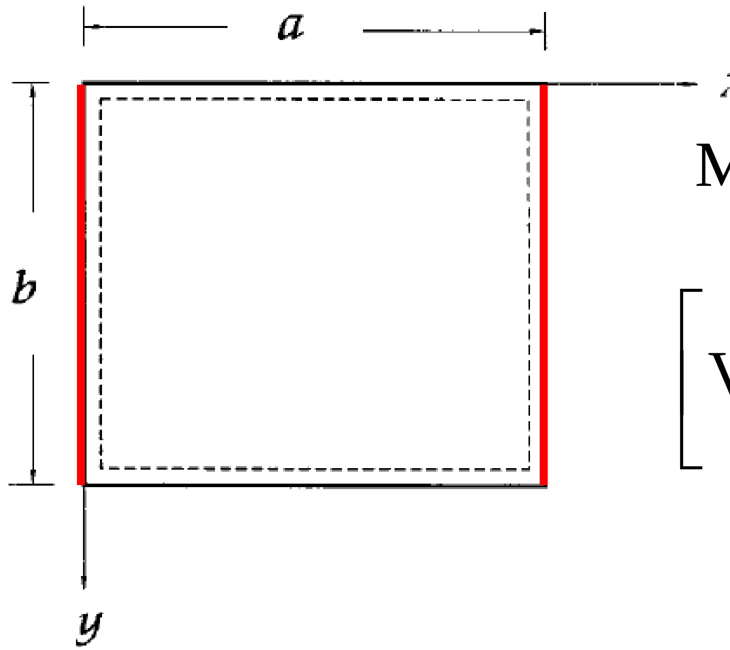
$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{k_0 w}{D} = \frac{q}{D}$$

Fundamental equations

Boundary conditions: free plate

$$M_{x|x=0} = 0$$

$$\left[V_x + \frac{\partial M_{xy}}{\partial y} \right]_{|x=0} = 0$$



$$M_{x|x=a} = 0$$

$$\left[V_x + \frac{\partial M_{xy}}{\partial y} \right]_{|x=a} = 0$$

$$M_x = -D \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} \right)$$

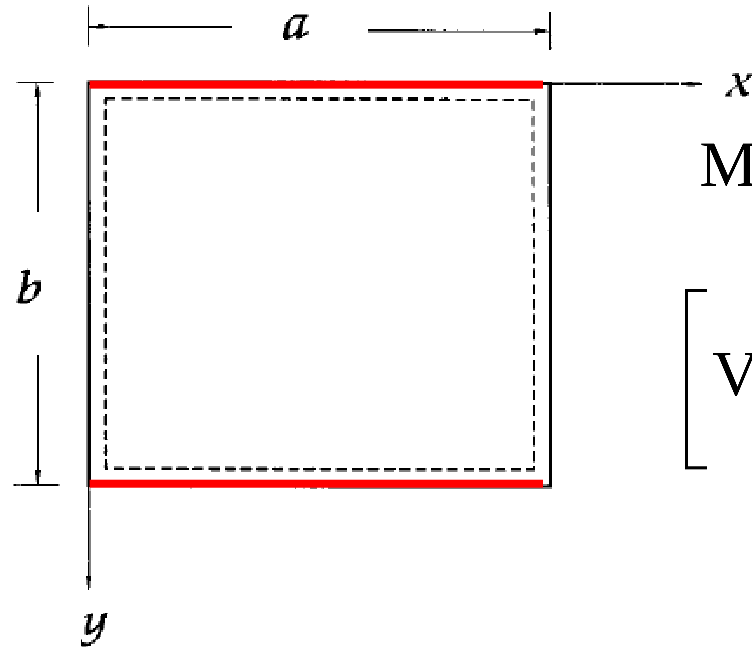
$$R_x = -D \cdot \frac{\partial}{\partial x} \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} \right) - D \cdot (1-\nu) \frac{\partial}{\partial y} \frac{\partial^2 w}{\partial x \partial y} = -D \cdot \left(\frac{\partial^3 w}{\partial x^3} + (2-\nu) \cdot \frac{\partial^3 w}{\partial x \partial y^2} \right)$$

Fundamental equations

Boundary conditions: free plate

$$M_{y|y=0} = 0$$

$$\left[V_y + \frac{\partial M_{yx}}{\partial x} \right]_{|y=0} = 0$$



$$M_{y|y=b} = 0$$

$$\left[V_y + \frac{\partial M_{yx}}{\partial x} \right]_{|y=b} = 0$$

$$M_y = -D \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2} \right)$$

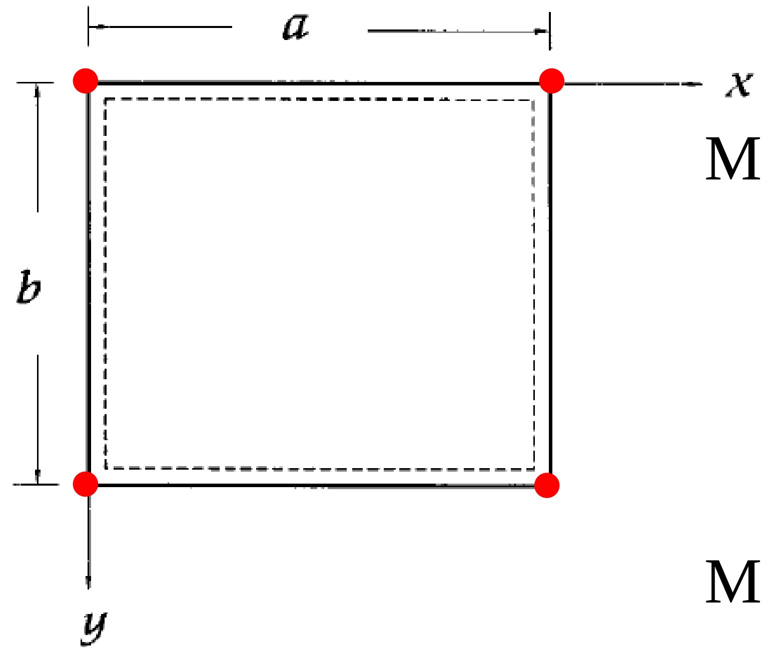
$$R_y = -D \cdot \left(\frac{\partial^3 w}{\partial y^3} + (2-\nu) \cdot \frac{\partial^3 w}{\partial x^2 \partial y} \right)$$

Fundamental equations

Boundary conditions: free plate

$$M_{xy|x=0,y=0} = 0$$

$$M_{xy|x=a,y=0} = 0$$



$$M_{xy|x=0,y=b} = 0$$

$$M_{xy|x=a,y=b} = 0$$

$$M_{xy} = M_{yx} = -(1-\nu)\cdot D\cdot\frac{\partial^2 w}{\partial x\partial y}$$

Overview

- ✓ Summary about the elastic theory of (thin) plates;
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Finite differences

Fundamental ideas

Any function f(x) can be represented by a "Taylor series", which is an *infinite sum* of terms that are calculated from the values of the function's derivatives at a single point.

Hence, any function can be *locally approximated* by its derivatives at that point.

$$f(x) = f(x_0 + \Delta x) = \underbrace{f(x_0) + f'(x_0)\cdot \Delta x + \frac{f''(x_0)}{2!}\cdot \Delta x^2}_{\text{Local approximation}} + \underbrace{o\left(|\Delta x|^2\right)}_{\text{Residual Error }(\Delta f)}$$

Local approximation *Residual Error (Δf)*

Example: *approximation of f(x)=sin x around x_0=0.*

$$f(x) = \sin x \qquad f(0) = 0$$

$$f'(x) = \cos x \qquad f'(0) = 1$$

$$f''(x) = -\sin x \qquad f''(0) = 0$$

$$x = \Delta x = 0.1 \qquad \Delta f(0) = 1.67\cdot 10^{-4}$$

$$\sin(0.1) = 0.099833$$

$$\sin(0) + \cos(0)\cdot 0.1 = 0.100000$$

$$x = \Delta x = 0.01 \qquad \Delta f(0) = 1.67\cdot 10^{-7}$$

$$\sin(0.01) = 0.010000$$

$$\sin(0) + \cos(0)\cdot 0.01 = 0.010000$$

Finite differences

Central difference scheme

$$f(x_0+\Delta x) \approx f(x_0) + f'(x_0)\cdot\Delta x + \frac{f''(x_0)}{2!}\cdot\Delta x^2 \qquad (1)$$

$$f(x_0-\Delta x) \approx f(x_0) - f'(x_0)\cdot\Delta x + \frac{f''(x_0)}{2!}\cdot\Delta x^2 \qquad (2)$$

An approximate expression of the first derivative can be obtained by *subtracting* (2) from (1).

$$f(x_0+\Delta x) - f(x_0-\Delta x) \approx 2\cdot f'(x_0)\cdot\Delta x$$

$$\boxed{f'(x_0) \approx \frac{f(x_0+\Delta x) - f(x_0-\Delta x)}{2\cdot\Delta x}}$$

An approximate expression of the second derivative can be obtained by *adding* (1) and (2).

$$f(x_0+\Delta x) + f(x_0-\Delta x) \approx 2\cdot f(x_0) + f''(x_0)\cdot\Delta x^2$$

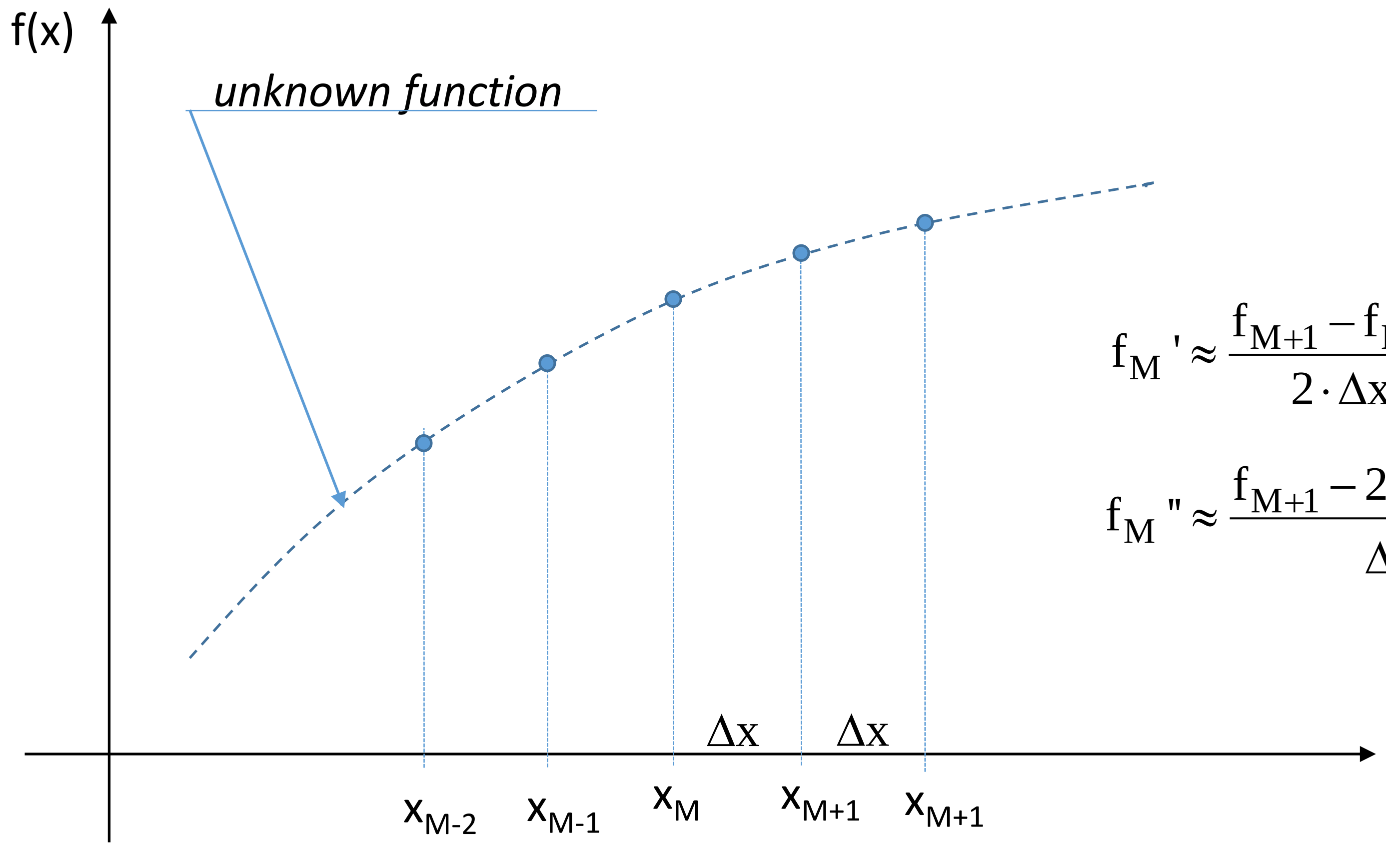
$$\boxed{f''(x_0) \approx \frac{f(x_0+\Delta x) - 2\cdot f(x_0) + f(x_0-\Delta x)}{\Delta x^2}}$$

Finite differences

Discretisation and approximation of derivatives based on the function values

$$f'(x_0) \approx \frac{f(x_0 + \Delta x) - f(x_0 - \Delta x)}{2 \cdot \Delta x}$$

$$f''(x_0) \approx \frac{f(x_0 + \Delta x) - 2 \cdot f(x_0) + f(x_0 - \Delta x)}{\Delta x^2}$$



$$f_M' \approx \frac{f_{M+1} - f_{M-1}}{2 \cdot \Delta x}$$

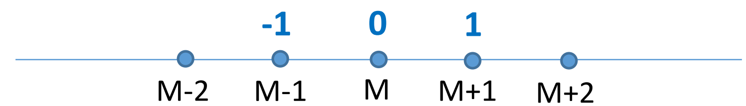
$$f_M'' \approx \frac{f_{M+1} - 2f_M + f_{M-1}}{\Delta x^2}$$

Finite differences

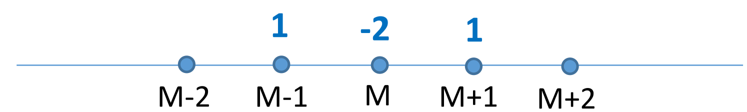
Approximation of higher-order derivatives

$$f_M$$

$$f_M' \approx \frac{f_{M+1} - f_{M-1}}{2 \cdot \Delta x}$$

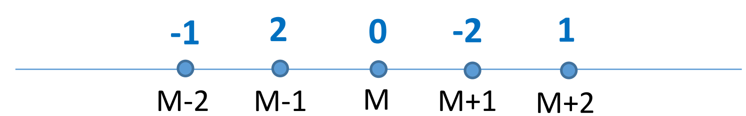


$$f_M'' \approx \frac{f_{M+1} - 2f_M + f_{M-1}}{\Delta x^2}$$

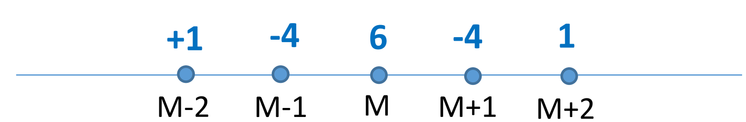


$$f_M''' = (f_M'')' \approx \left(\frac{f_{M+1} - 2f_M + f_{M-1}}{\Delta x^2} \right)' = \frac{1}{\Delta x^2} \cdot \left(\frac{f_{M+2} - f_M}{2 \cdot \Delta x} - 2 \cdot \frac{f_{M+1} - f_{M-1}}{2 \cdot \Delta x} + \frac{f_M - f_{M-2}}{2 \cdot \Delta x} \right)$$

$$f_M''' = \frac{f_{M+2} - 2 \cdot f_{M+1} + 2 \cdot f_{M-1} - f_{M+2}}{2 \cdot \Delta x^3}$$



$$f_M'''' = (f_M'')'' \approx \left(\frac{f_{M+1} - 2f_M + f_{M-1}}{\Delta x^2} \right)'' = \frac{f_{M+2} - 4 \cdot f_{M+1} + 6 \cdot f_M - 4 \cdot f_{M-1} + f_{M-2}}{\Delta x^4}$$



Overview

- ✓ Summary about the elastic theory of (thin) plates;
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Finite difference

Conceptual approach

Differential formulation

Field equation

$$\frac{\partial^4 w}{\partial x^4} + 2 \cdot \frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{k_0 w}{D} = \frac{q}{D}$$

Boundary conditions

$$M_x = -D \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} \right) = 0$$

$$R_x = -D \cdot \left(\frac{\partial^3 w}{\partial x^3} + (2-\nu) \cdot \frac{\partial^3 w}{\partial x \partial y^2} \right) = 0$$

Main unknown

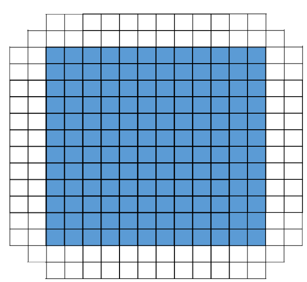
$$w = w(x, y)$$

Finite Difference Discretization

Algebraic conversion

Field equations

Boundary conditions



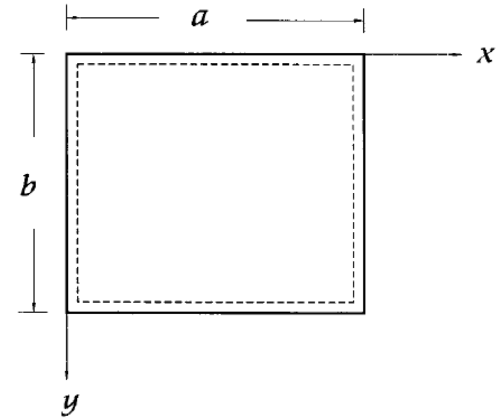
Main unknowns

$$w_{M,N}$$

Finite difference

Central difference expressions

$$w_{M,N} = w\left(x_M, y_N\right)$$

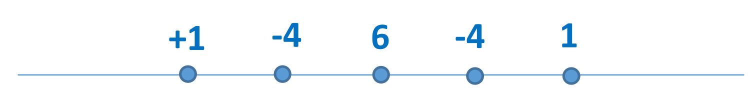


$$\left.\frac{\partial w}{\partial x}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M+1,N} - w_{M-1,N}}{2\cdot\Delta}$$

$$\left.\frac{\partial^2 w}{\partial x^2}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M+1,N} - 2\cdot w_{M,N} + w_{M-1,N}}{\Delta^2}$$

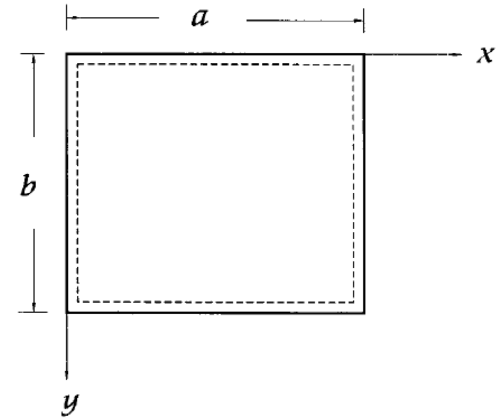
$$\left.\frac{\partial^3 w}{\partial x^3}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M+2,N} - 2\cdot w_{M+1,N} + 2\cdot w_{M-1,N} - w_{M-2,N}}{2\cdot\Delta^3}$$

$$\left.\frac{\partial^4 w}{\partial x^4}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M+2,N} - 4\cdot w_{M+1,N} + 6\cdot w_{M,N} - 4\cdot w_{M-1,N} + w_{M-2,N}}{\Delta^4}$$



Finite difference

Central difference expressions



$$w_{M,N} = w\left(x_M, y_N\right)$$

$$\left.\frac{\partial w}{\partial y}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M,N+1} - w_{M,N-1}}{2 \cdot \Delta}$$

$$\left.\frac{\partial^2 w}{\partial y^2}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M,N+1} - 2 \cdot w_{M,N} + w_{M,N-1}}{\Delta^2}$$

$$\left.\frac{\partial^3 w}{\partial y^3}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M,N+2} - 2 \cdot w_{M,N+1} + 2 \cdot w_{M,N-1} - w_{M,N-2}}{2 \cdot \Delta^3}$$

$$\left.\frac{\partial^4 w}{\partial y^4}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M,N+2} - 4 \cdot w_{M,N+1} + 6 \cdot w_{M,N} - 4 \cdot w_{M,N-1} + w_{M,N-2}}{\Delta^4}$$

Finite difference

Central difference expressions

$$\left.\frac{\partial^2 w}{\partial x \partial y}\right|_{\substack{x=x_M\\ y=y_N}} = \left.\frac{\partial}{\partial y}\frac{\partial w}{\partial x}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M+1,N+1} - w_{M+1,N-1} - w_{M-1,N+1} + w_{M-1,N-1}}{4\cdot\Delta^2}$$

$$\left.\frac{\partial^4 w}{\partial x^2 \partial y^2}\right|_{\substack{x=x_M\\ y=y_N}} = \left.\frac{\partial^2}{\partial x^2}\frac{\partial^2 w}{\partial y^2}\right|_{\substack{x=x_M\\ y=y_N}} =$$

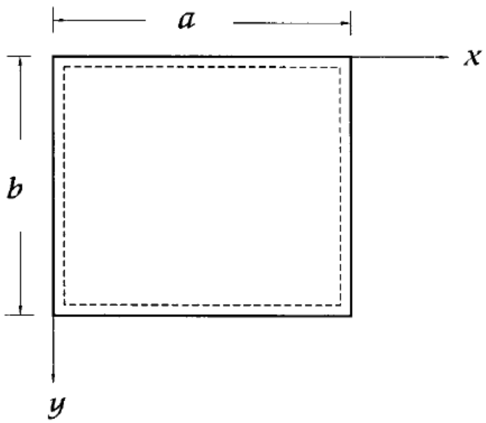
$$= \frac{w_{M+1,N+1} - 2w_{M+1,N} + w_{M+1,N-1} - 2\cdot w_{M,N+1} + 4\cdot w_{M,N+1} - 2\cdot w_{M,N-1} + w_{M-1,N+1} - 2w_{M-1,N} + w_{M-1,N-1}}{\Delta^4}$$

$$\left.\frac{\partial^3 w}{\partial x \partial y^2}\right|_{\substack{x=x_M\\ y=y_N}} = \left.\frac{\partial}{\partial x}\frac{\partial^2 w}{\partial y^2}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M+1,N+1} - w_{M-1,N+1} - 2\cdot w_{M+1,N} + 2\cdot w_{M-1,N} + w_{M+1,N-1} - w_{M-1,N-1}}{2\cdot\Delta^3}$$

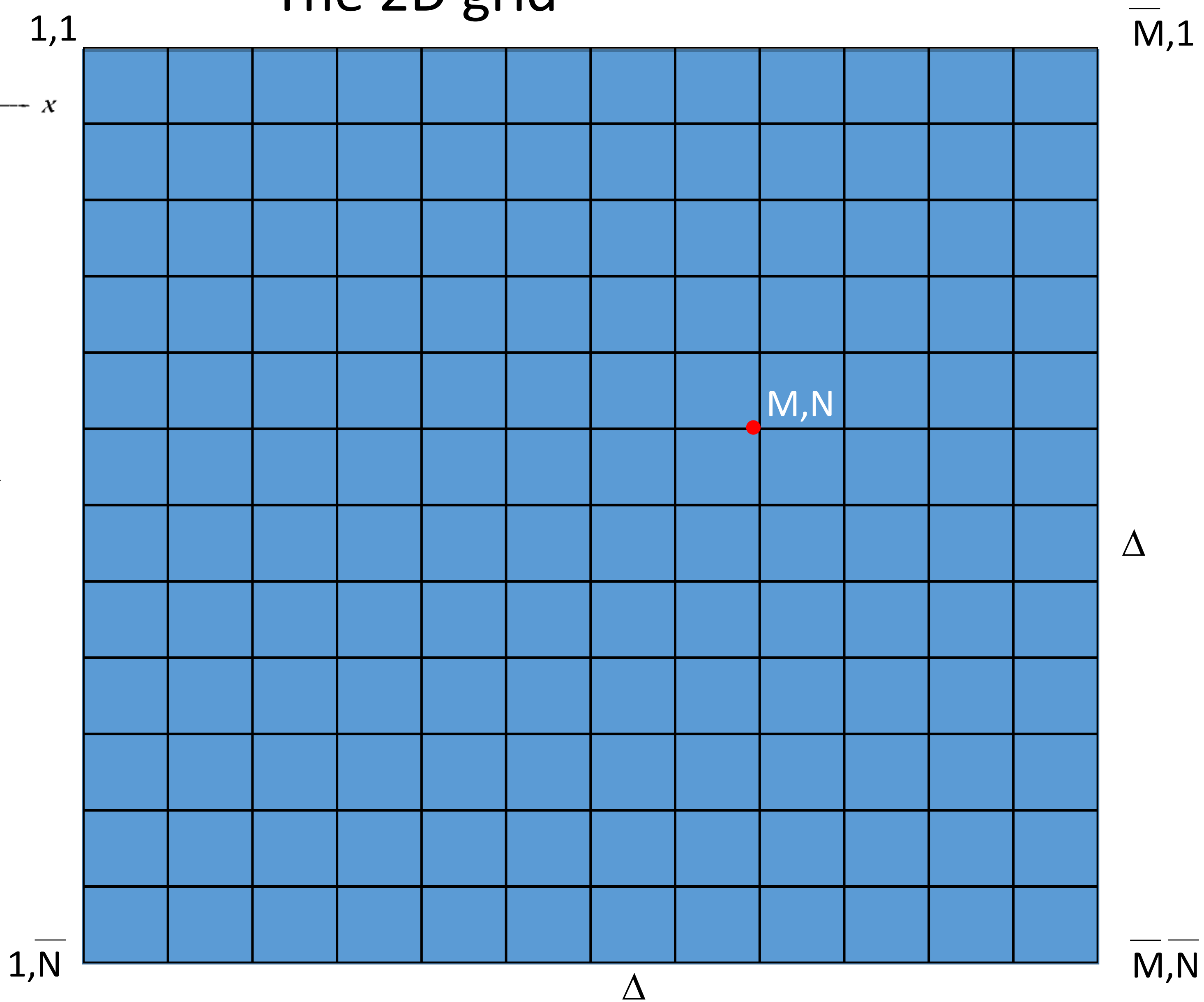
$$\left.\frac{\partial^3 w}{\partial x^2 \partial y}\right|_{\substack{x=x_M\\ y=y_N}} = \left.\frac{\partial}{\partial y}\frac{\partial^2 w}{\partial x^2}\right|_{\substack{x=x_M\\ y=y_N}} = \frac{w_{M+1,N+1} - w_{M+1,N-1} - 2\cdot w_{M,N+1} + 2\cdot w_{M,N-1} + w_{M-1,N+1} - w_{M-1,N-}}{2\Delta^3}$$

Finite difference

The 2D grid



$$\bar{M} = \frac{a}{\Delta} + 1 \qquad \bar{N} = \frac{b}{\Delta} + 1$$

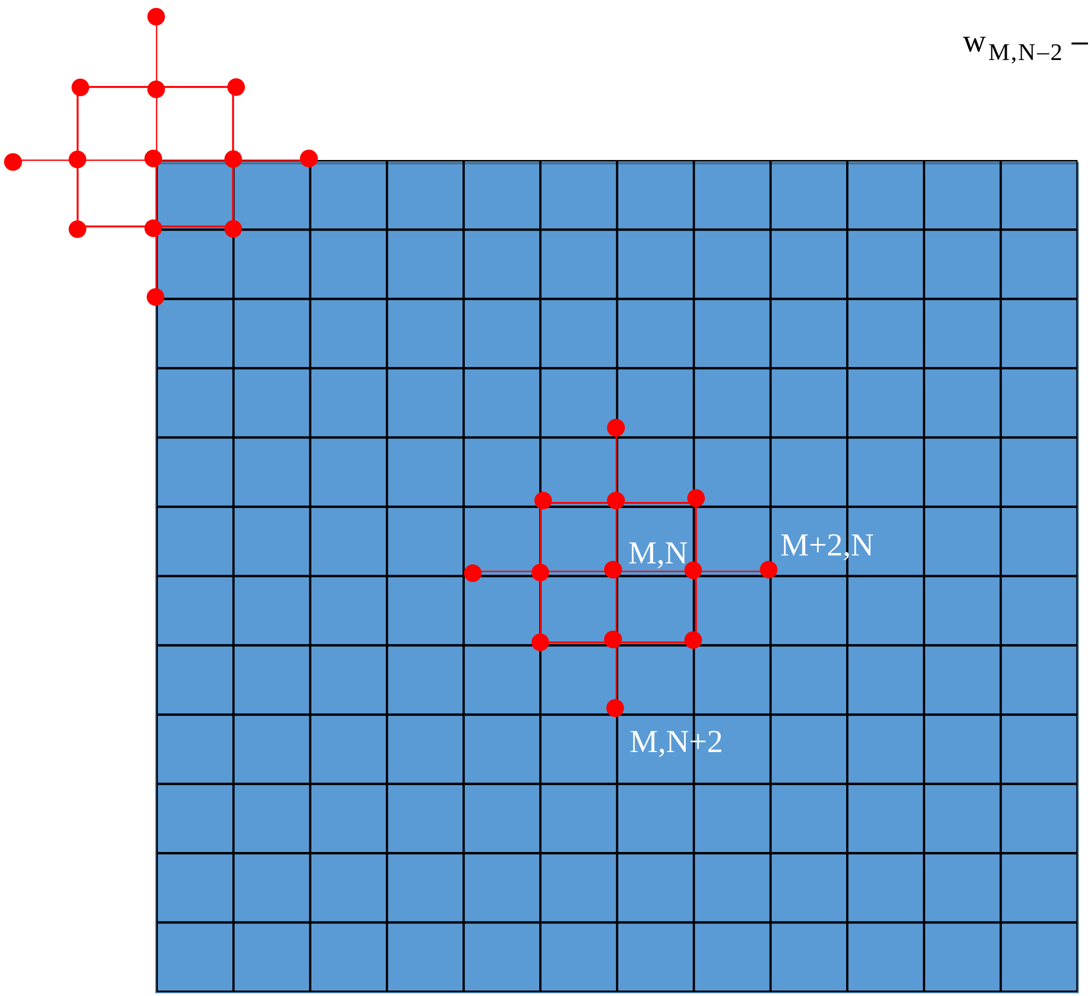


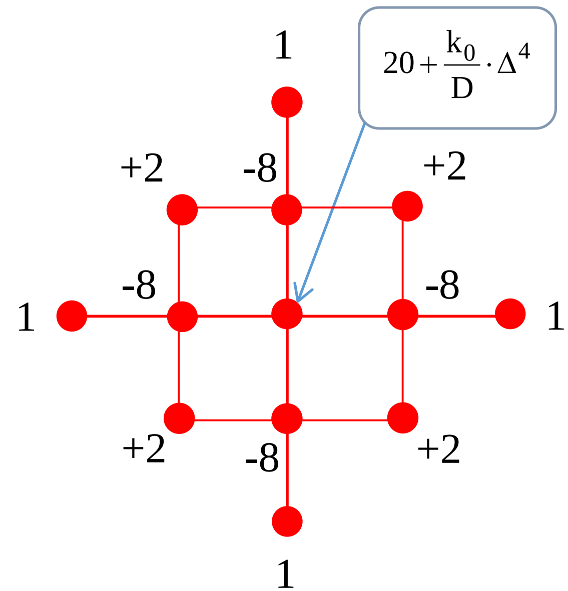
Finite difference

Central difference expressions

$$\frac{\partial^4 w}{\partial x^4} + 2\cdot\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} + \frac{k_0 w}{D} = \frac{q}{D}$$

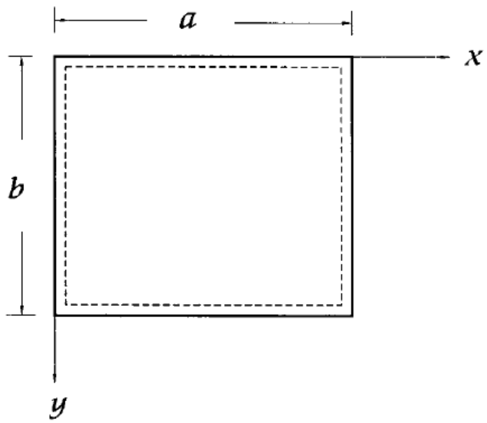
$$w_{M+2,N} +$$
$$2\cdot w_{M+1,N+1} - 8\cdot w_{M+1,N} + 2\cdot w_{M+1,N-1} +$$
$$w_{M,N-2} - 8\cdot w_{M,N-1} + \left(20 + \frac{k_0}{D}\cdot\Delta^4\right)\cdot w_{M,N} - 8\cdot w_{M,N+1} + w_{M,N+2} +$$
$$2\cdot w_{M-1,N+1} - 8\cdot w_{M-1,N} + 2\cdot w_{M-1,N-1} +$$
$$w_{M-2,N} =$$
$$= \frac{q_{M,N}}{D}\cdot\Delta^4$$





Finite difference

Central difference expressions



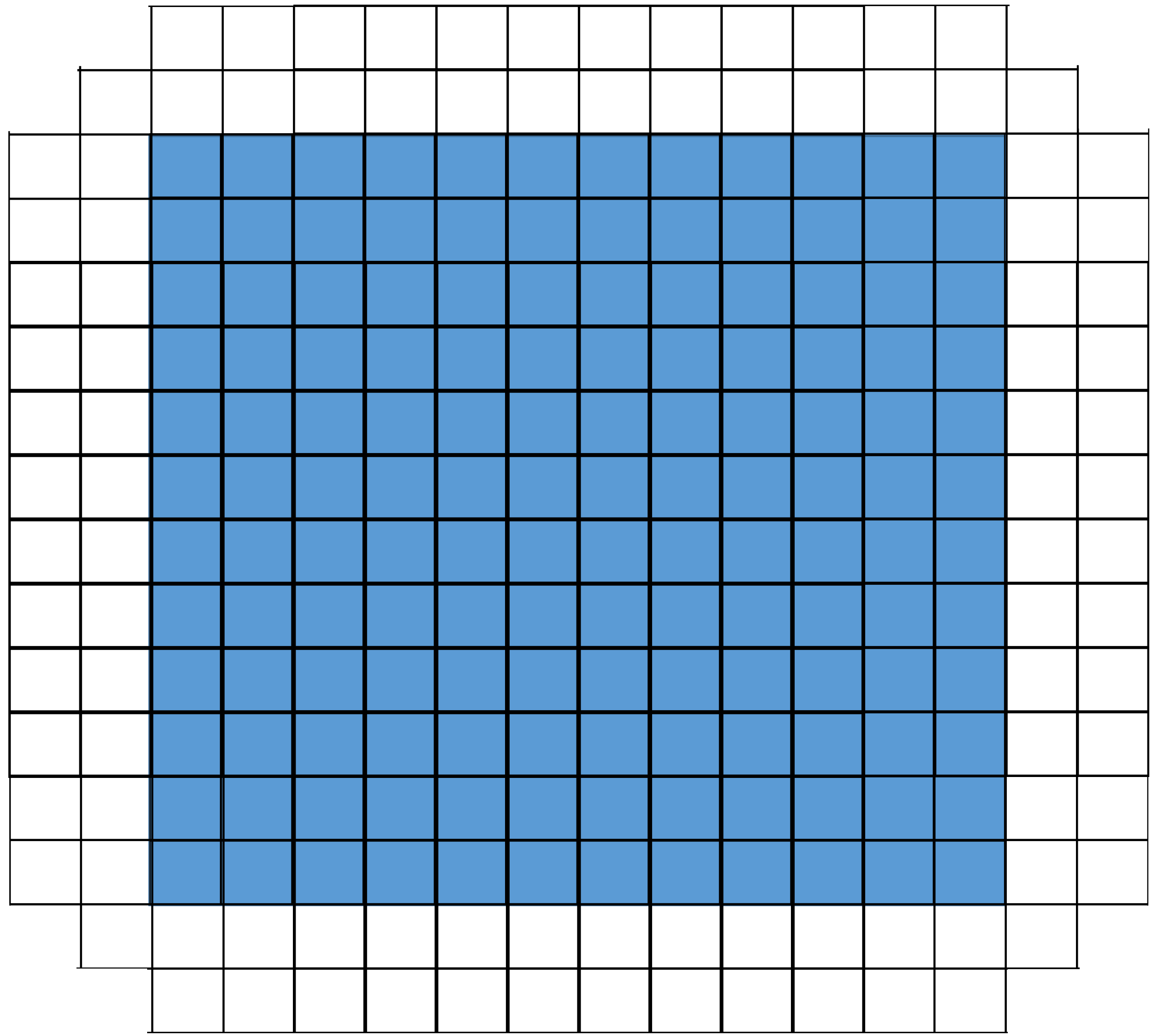
$$\bar{M} = \frac{a}{\Delta} + 1 \qquad \bar{N} = \frac{b}{\Delta} + 1$$

Total number of unknowns:

$$\bar{M} \cdot \bar{N} + 4 \cdot \left(\bar{M} + \bar{N} + 1 \right)$$

Total number of equations:

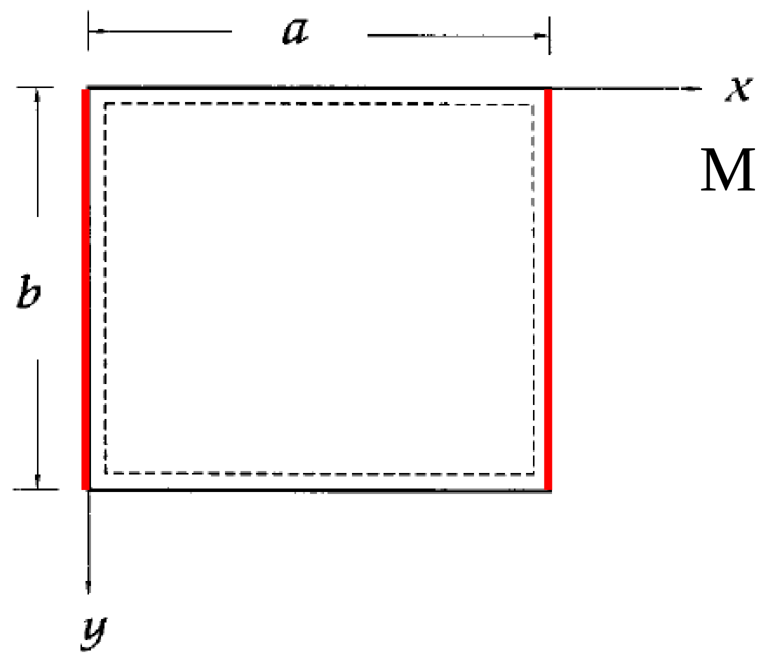
$$\bar{M} \cdot \bar{N}$$



Finite difference

Boundary conditions: free side

$$M_{x|x=0} = 0$$

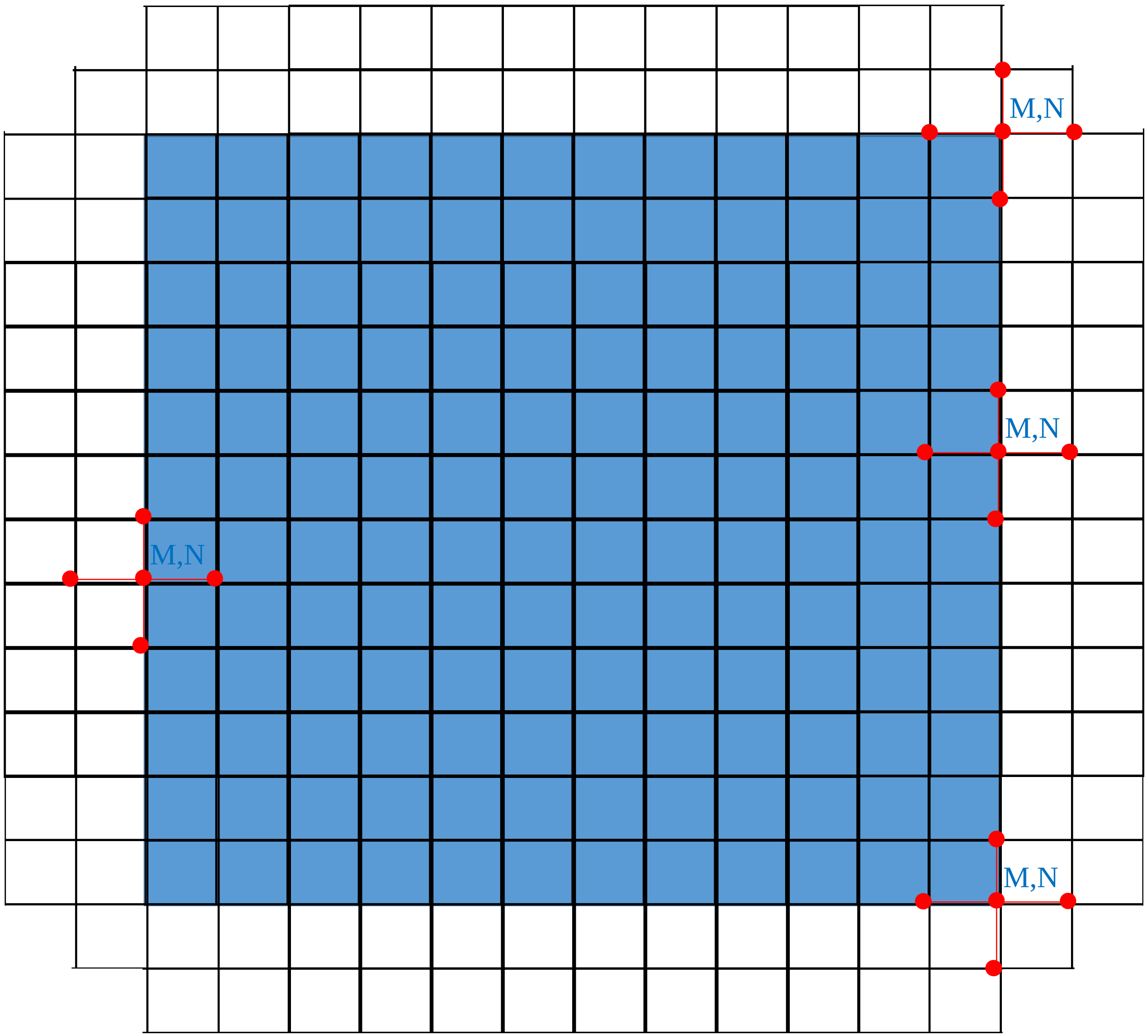


$$M_{x|x=a} = 0$$

$$M_x = -D \cdot \left(\frac{\partial^2 w}{\partial x^2} + \nu \cdot \frac{\partial^2 w}{\partial y^2} \right)$$

Finite difference

Central difference expressions



$$-D\cdot\left(\frac{\partial^2 w}{\partial x^2}+\nu\cdot\frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{x=a}=0$$

$$\begin{aligned}&\nu\cdot w_{M,N+1}\\ +w_{M-1,N}-2\cdot(1+\nu)\cdot w_{M,N}+&w_{M+1,N}\\ &+\nu\cdot w_{M,N-1}\\ &=0\end{aligned}$$

$$-D\cdot\left(\frac{\partial^2 w}{\partial x^2}+\nu\cdot\frac{\partial^2 w}{\partial y^2}\right)\Bigg|_{x=0}=0$$

Total number of new equations:

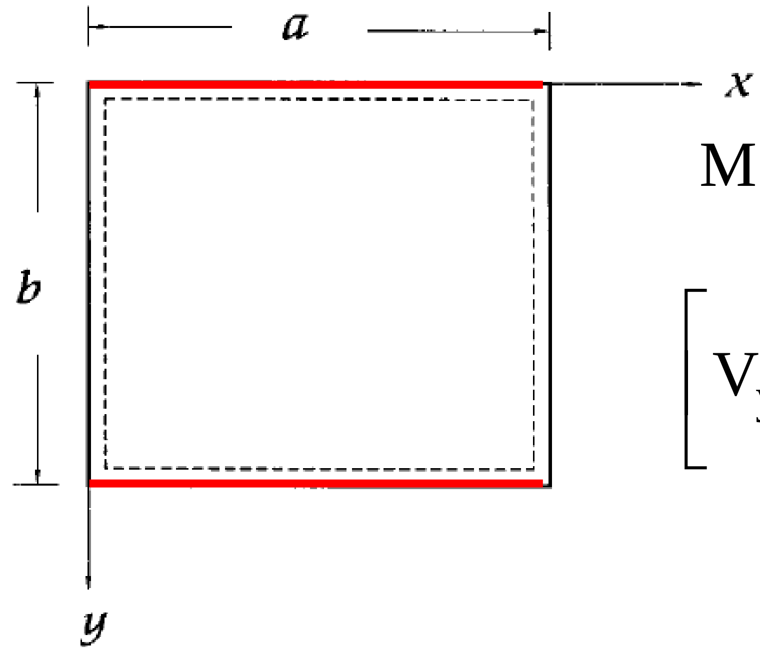
$$2\cdot\bar{N}$$

Finite Difference

Boundary conditions: free side

$$M_{y|y=0} = 0$$

$$\left[V_y + \frac{\partial M_{yx}}{\partial x}\right]_{|y=0} = 0$$



$$M_{y|y=b} = 0$$

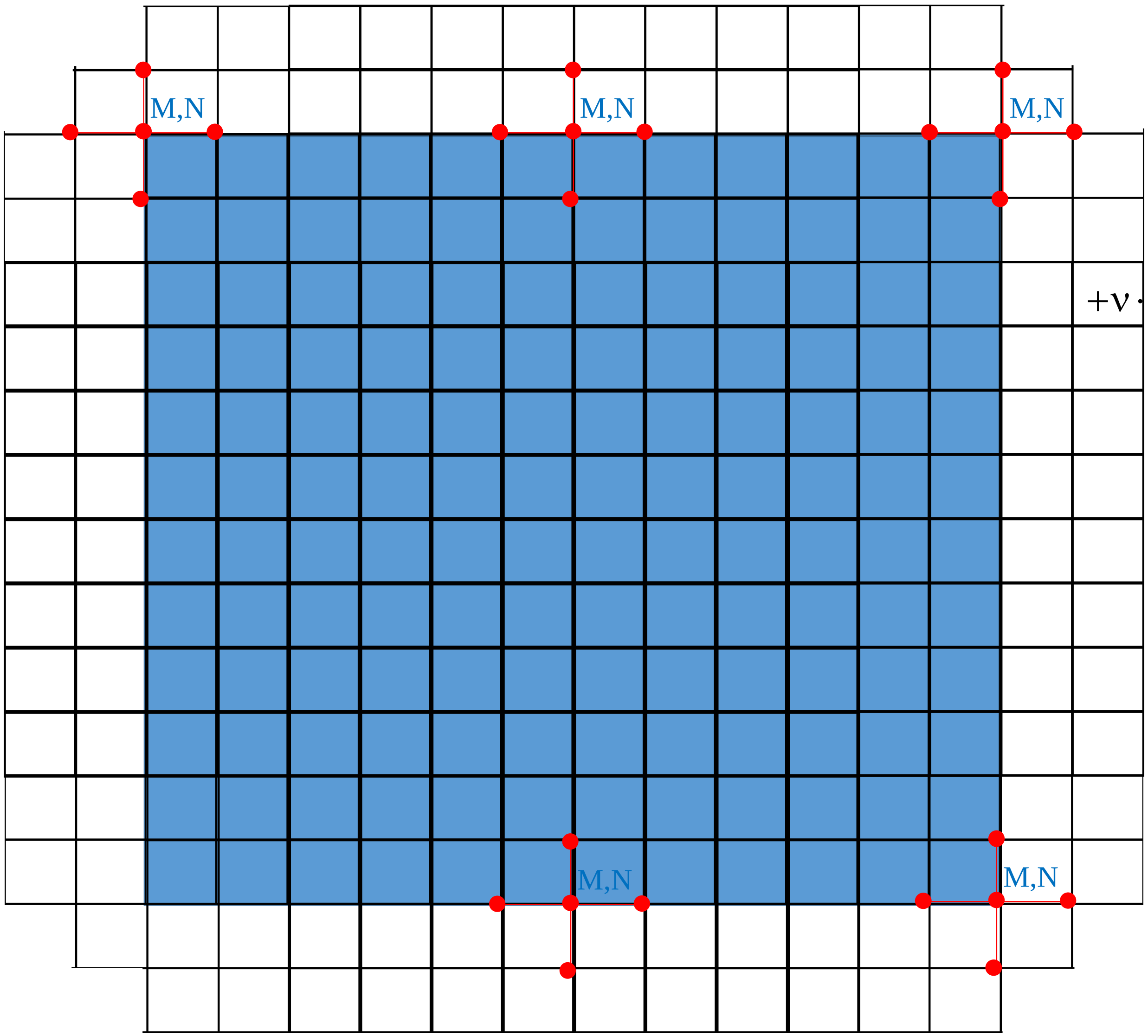
$$\left[V_y + \frac{\partial M_{yx}}{\partial x}\right]_{|y=b} = 0$$

$$M_y = -D \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2} \right)$$

$$R_y = -D \cdot \left(\frac{\partial^3 w}{\partial y^3} + (2 - \nu) \cdot \frac{\partial^3 w}{\partial x^2 \partial y} \right)$$

Finite difference

Central difference expressions



$$-D \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2} \right) \Bigg|_{y=b} = 0$$

$$w_{M+1,N}$$
$$+\nu \cdot w_{M,N-1} - 2 \cdot (1+\nu) \cdot w_{M,N} + \nu \cdot w_{M,N+1}$$
$$+w_{M-1,N}$$
$$= 0$$

$$-D \cdot \left(\frac{\partial^2 w}{\partial y^2} + \nu \cdot \frac{\partial^2 w}{\partial x^2} \right) \Bigg|_{y=0} = 0$$

Total number of <u>new</u> <u>equations</u>:

$$2 \cdot \bar{M}$$

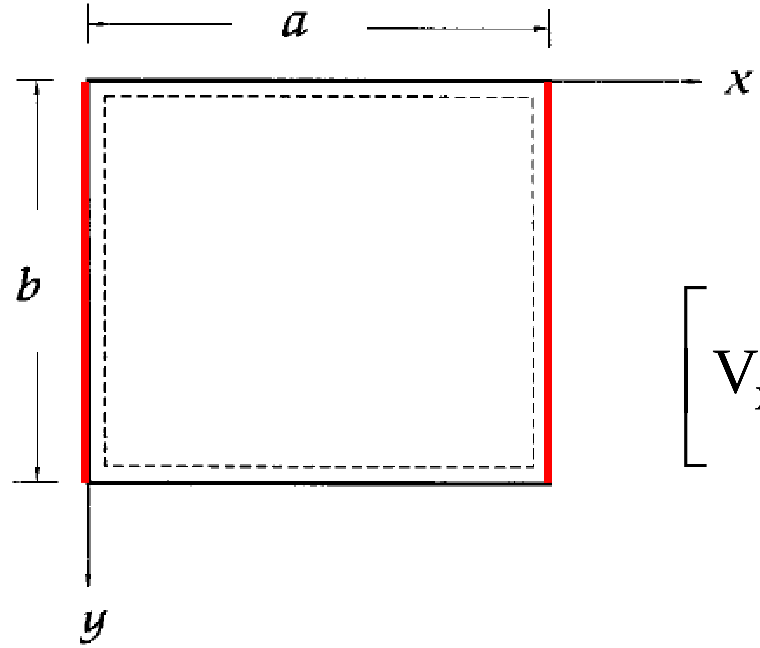
Total number of <u>equations</u>:

$$\bar{M} \cdot \bar{N} + 2 \cdot \left(\bar{M} + \bar{N} \right)$$

Finite difference

Boundary conditions: free side

$$\left[V_x + \frac{\partial M_{xy}}{\partial y}\right]_{|x=0} = 0$$



$$\left[V_x + \frac{\partial M_{xy}}{\partial y}\right]_{|x=a} = 0$$

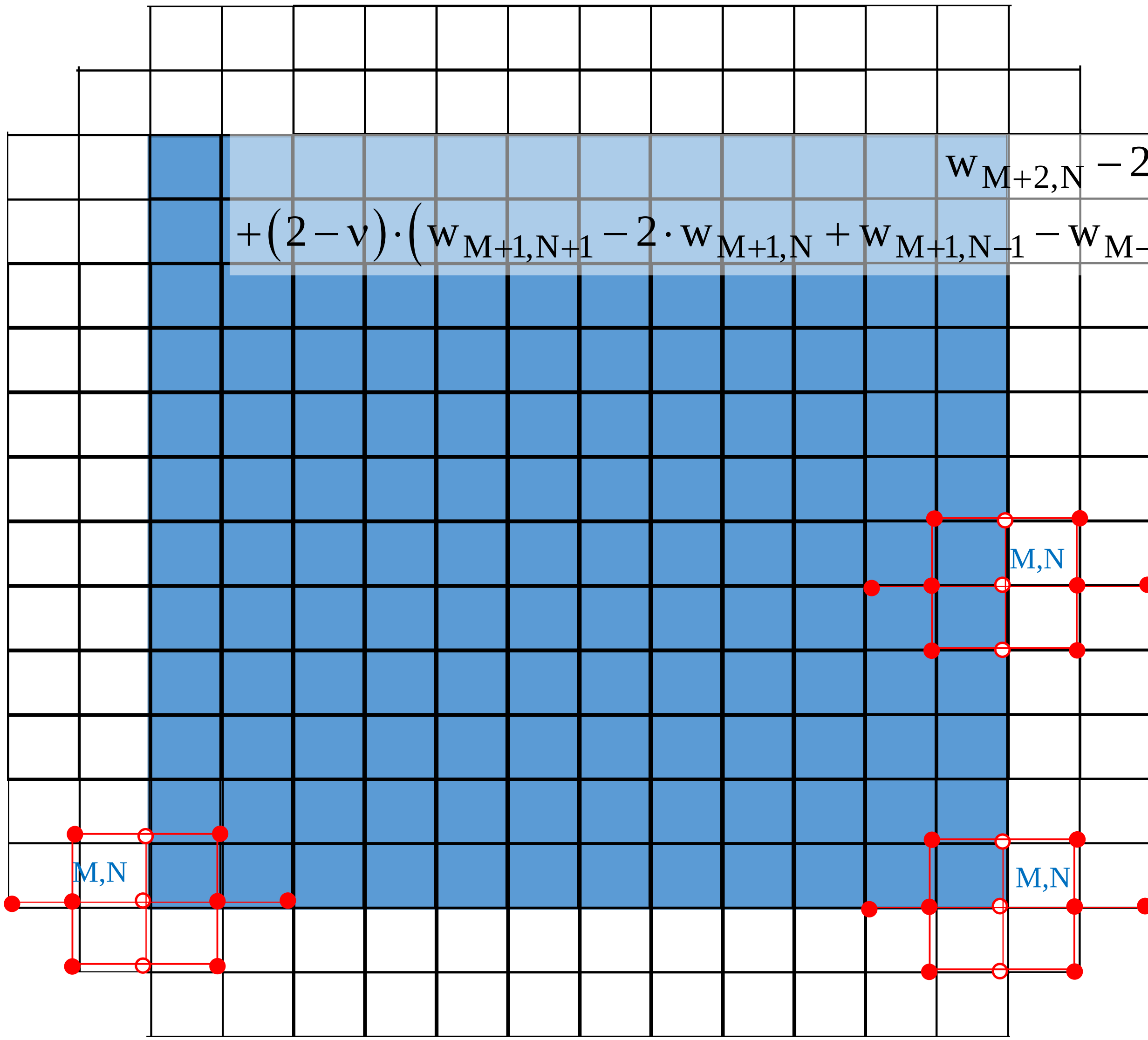
$$R_x = -D\cdot\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right) - D\cdot(1-\nu)\frac{\partial}{\partial y}\frac{\partial^2 w}{\partial x \partial y} = -D\cdot\left(\frac{\partial^3 w}{\partial x^3}+(2-\nu)\cdot\frac{\partial^3 w}{\partial x \partial y^2}\right)$$

Finite difference

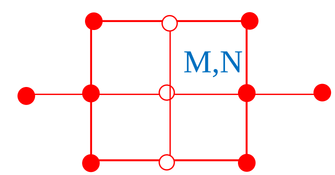
Central difference expressions

$$-D\cdot\left(\frac{\partial^3 w}{\partial x^3}+(2-\nu)\cdot\frac{\partial^3 w}{\partial x\partial y^2}\right)_{x=a}=0$$

$$w_{M+2,N}-2\cdot w_{M+1,N}+2\cdot w_{M-1,N}-w_{M-2,N}+$$
$$+(2-\nu)\cdot\left(w_{M+1,N+1}-2\cdot w_{M+1,N}+w_{M+1,N-1}-w_{M-1,N+1}+2\cdot w_{M-1,N}-w_{M-1,N-1}\right)=0$$



$$-D\cdot\left(\frac{\partial^3 w}{\partial x^3}+(2-\nu)\cdot\frac{\partial^3 w}{\partial x\partial y^2}\right)_{x=0}=0$$



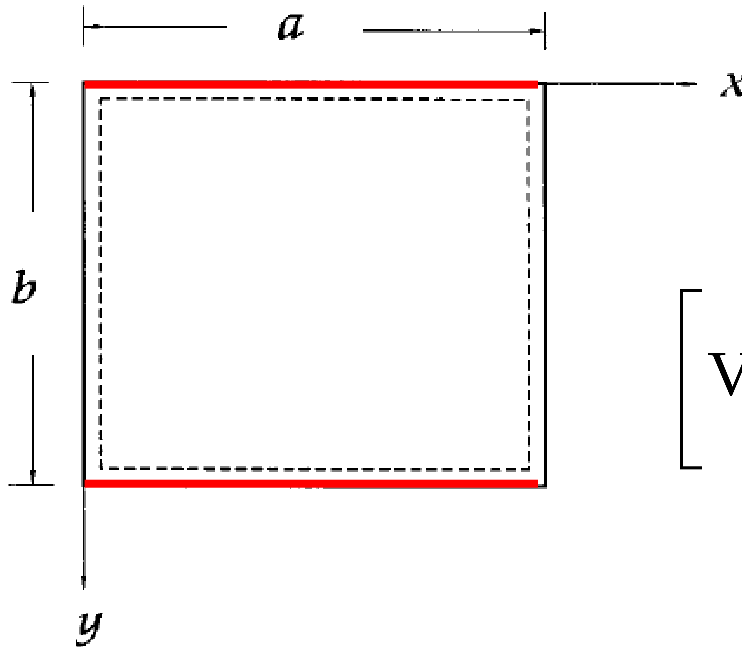
Total number of <u>new</u> <u>equations</u>:

$$2\cdot\bar{N}$$

Finite Difference

Boundary conditions: free side

$$\left[V_y + \frac{\partial M_{yx}}{\partial x}\right]_{|y=0} = 0$$



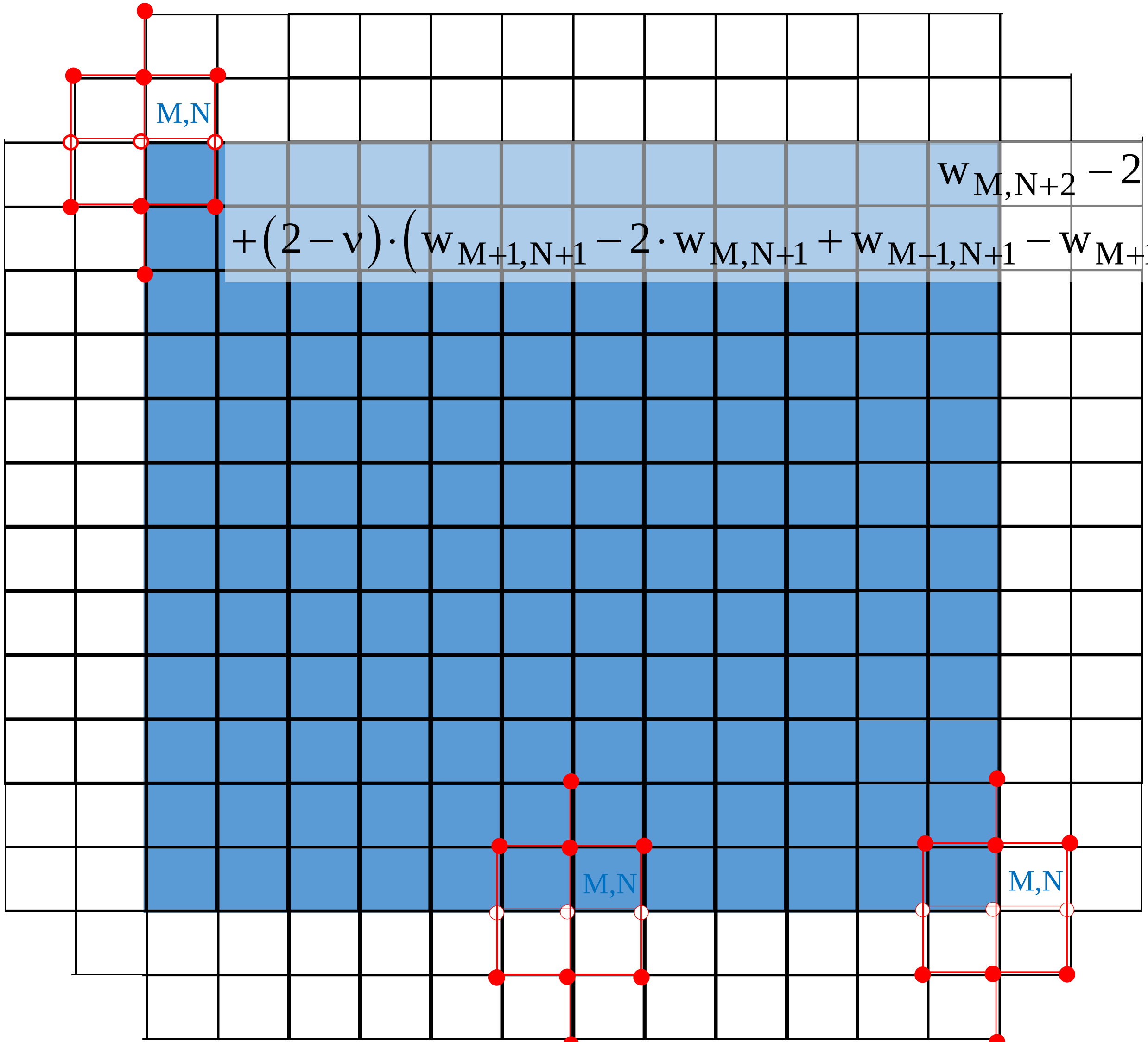
$$\left[V_y + \frac{\partial M_{yx}}{\partial x}\right]_{|y=b} = 0$$

$$R_y = -D \cdot \left(\frac{\partial^3 w}{\partial y^3} + (2-\nu) \cdot \frac{\partial^3 w}{\partial x^2 \partial y} \right)$$

Finite difference

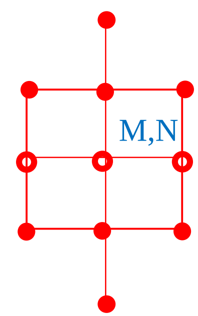
Central difference expressions

$$-D\cdot\left(\frac{\partial^3 w}{\partial y^3}+(2-\nu)\cdot\frac{\partial^3 w}{\partial x^2\partial y}\right)_{y=b}=0$$



$$w_{M,N+2}-2\cdot w_{M,N+1}+2\cdot w_{M,N-1}-w_{M,N-2}+$$
$$+(2-\nu)\cdot\left(w_{M+1,N+1}-2\cdot w_{M,N+1}+w_{M-1,N+1}-w_{M+1,N-1}+2\cdot w_{M,N-1}-w_{M-1,N-1}\right)=0$$

$$-D\cdot\left(\frac{\partial^3 w}{\partial y^3}+(2-\nu)\cdot\frac{\partial^3 w}{\partial x^2\partial y}\right)_{y=0}=0$$



Total number of <u>new</u> <u>equations</u>:

$$2\cdot\bar{M}$$

Total number of <u>equations</u>:

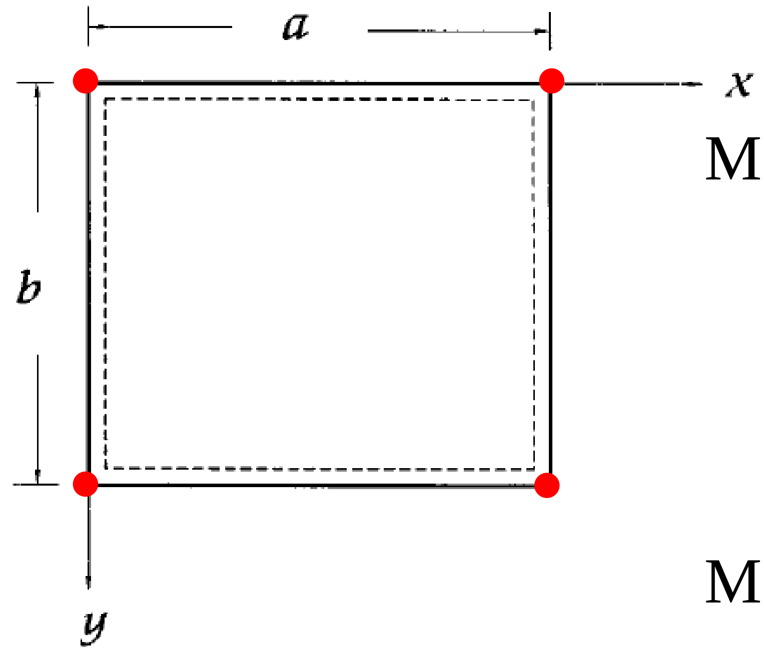
$$\bar{M}\cdot\bar{N}+4\cdot\left(\bar{M}+\bar{N}\right)$$

Fundamental equations

Boundary conditions: free plate

$$M_{xy|x=0,y=0} = 0$$

$$M_{xy|x=a,y=0} = 0$$



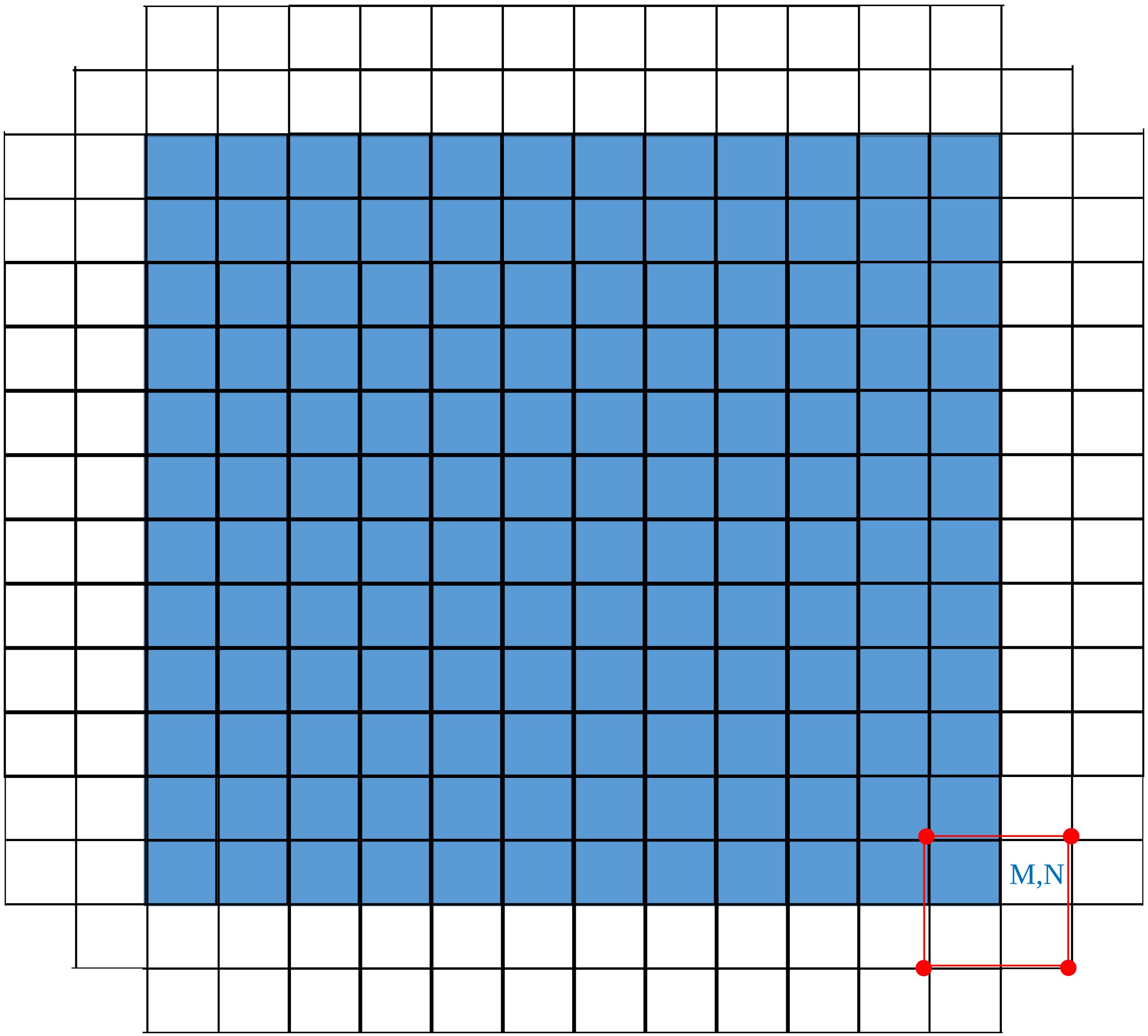
$$M_{xy|x=0,y=b} = 0$$

$$M_{xy|x=a,y=b} = 0$$

$$M_{xy} = M_{yx} = -(1-\nu)\cdot D\cdot\frac{\partial^2 w}{\partial x\partial y}$$

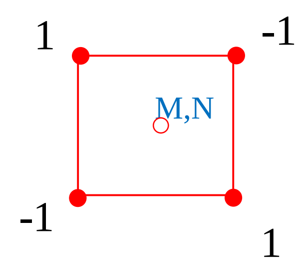
Finite difference

Central difference expressions



$$-\left(1-\nu\right)\cdot D\cdot\left.\frac{\partial^2 w}{\partial x\partial y}\right|_{\substack{x=a\\y=b}}=0$$

$$w_{M+1,N+1}-w_{M+1,N-1}+$$
$$-w_{M-1,N+1}+w_{M-1,N-1}=0$$



Total number of <u>new</u> <u>equations</u>:

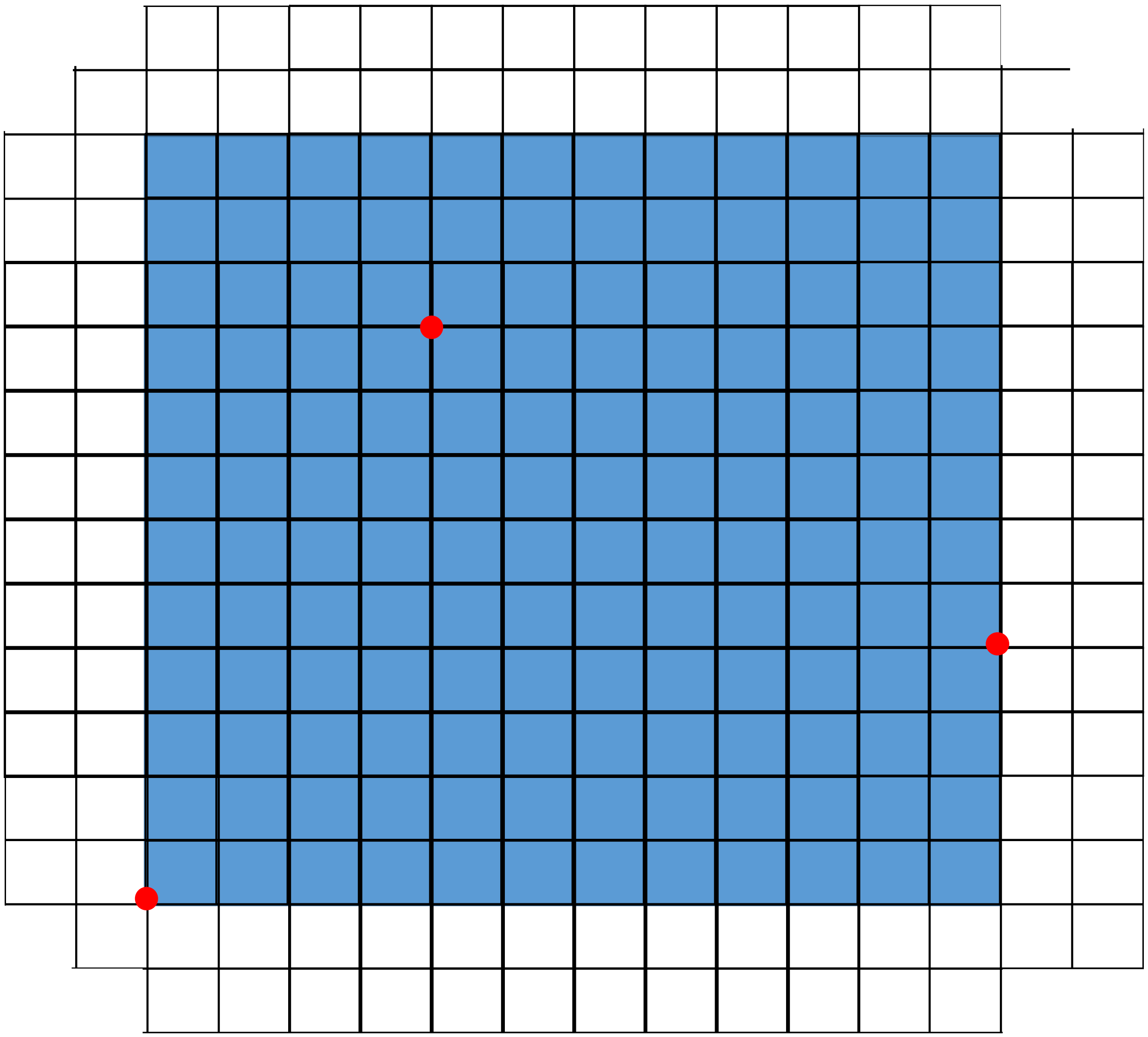
4

Total number of <u>equations</u>:

$\bar{M}\cdot\bar{N}+4\cdot\left(\bar{M}+\bar{N}+1\right)$

Finite difference

Toward a spreadsheet



Unknowns:

$w_{M,N}$

Total number of unknowns:

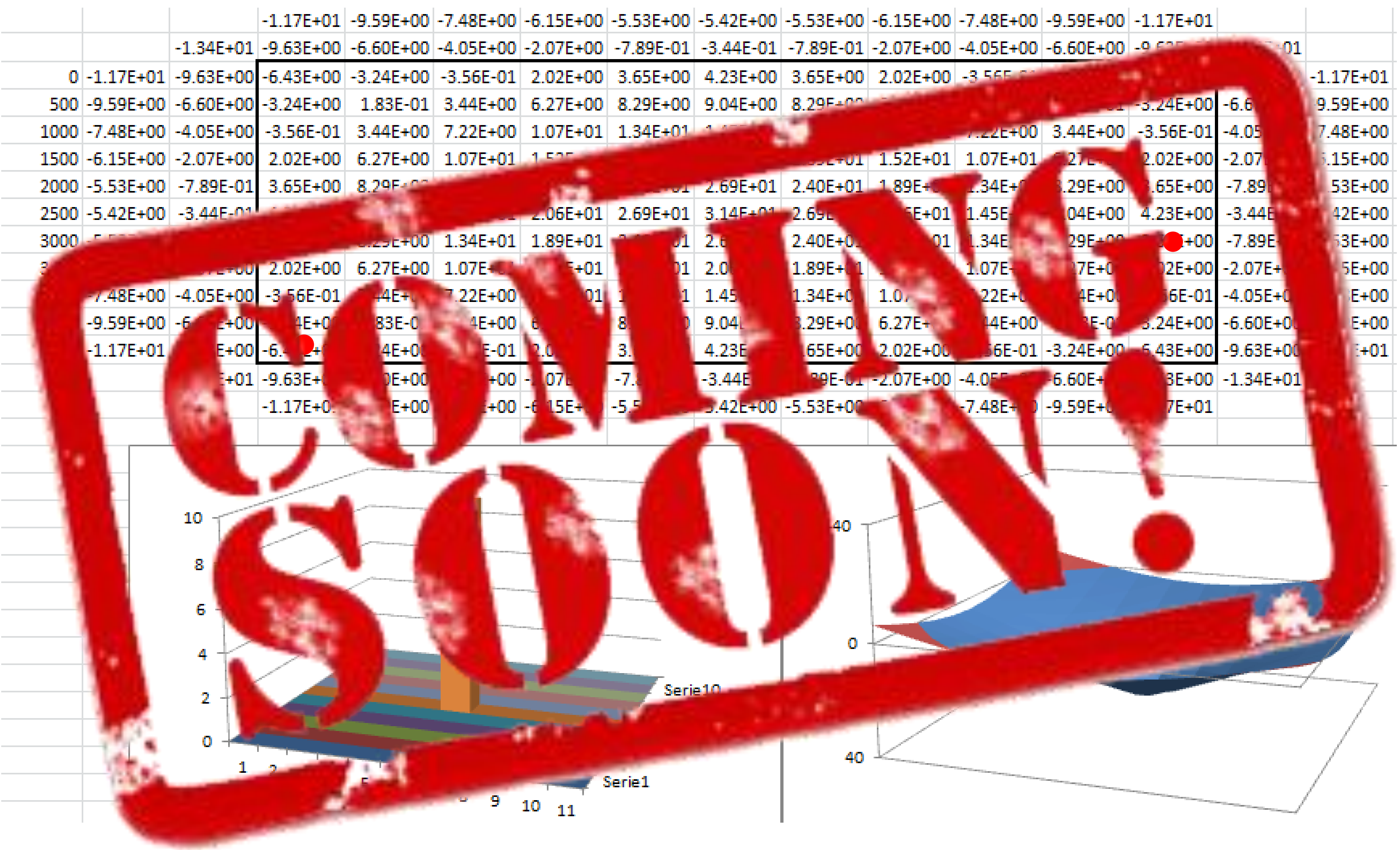
$$\bar{M} \cdot \bar{N} + 4 \cdot \left(\bar{M} + \bar{N} + 1\right)$$

Total number of equations:

$$\bar{M} \cdot \bar{N} + 4 \cdot \left(\bar{M} + \bar{N} + 1\right)$$

Overview

The final goal: a spreadsheet for the analysis of slabs on grade



The End

Thank you for your attention